(c) Letting $r \to 1^-$ in part (b), deduce that $|f(\xi)| \leq |\xi|$ for all ξ in U.

17. Let f be a function satisfying the conditions of Schwarz's Lemma (Prob. 16). Prove that if $|f(z_0)| = |z_0|$ for some nonzero z_0 in U, then f must be a function of the form $f(z) = e^{i\theta}z$ for some real θ. Show also that f must be of this form if $|f'(0)| = 1$.

5.7 THE POINT AT INFINITY

From our discussion of singularities in Sec. 5.6 we know that if a mapping is given by an analytic function possessing a pole, it carries points near that pole to indefinitely distant points. It must have occurred to the reader that one might define the value of f *at* the pole to be ∞. Before taking this plunge, however, we should be aware of all the ramifications. Let us look in detail at the behavior of $1/z$ near $z = 0$.

As $z \to 0$ along the positive real axis, $1/z$ goes to "plus infinity"; along the negative real axis, $1/z$ goes to "minus infinity"; and along the positive y-axis, $1/z$ goes to—what? "Minus i times infinity?" If we are to assign the symbol ∞ to $1/0$ we must realize that we are identifying all these "limits" as a single number; geometrically, we are speaking of *the point at infinity*, which can be reached, in a manner of speaking, by proceeding infinitely far along any direction in the complex plane.

It is somewhat enlightening to try to visualize the situation in the following manner: Consider the rays emanating from the origin in the complex plane to all be joined at their "ends," deforming the complex plane into something like an upside-down parachute with its lines tied together. In fact, such an image motivates the *stereographic projection* depicted in Fig. 5.5 wherein the xy-plane is mapped onto the surface of the three-dimensional unit sphere which has the xy-plane as its equatorial plane. The mapping is defined as follows: Starting from a point P in the plane, we draw a line through the north pole of the sphere. Then the point P is mapped to the point P', where this line intersects the sphere, as illustrated in Fig. 5.5.

Thus the open disk $|z| < 1$ is mapped onto the southern hemisphere, the circle $|z| = 1$ onto the equator, and the exterior $|z| > 1$ onto the northern hemisphere excluding the north pole. Notice that all distant points get mapped onto "arctic zones" near the north pole; the latter is, in fact, the actual limit of such points, in the topology of the sphere. It corresponds to the "point at ∞."

With this model in mind, we officially append "∞" to the set of complex numbers, calling the resulting collection the *extended complex plane*. A neighborhood of ∞ is any set of the form $|z| > M$, and a

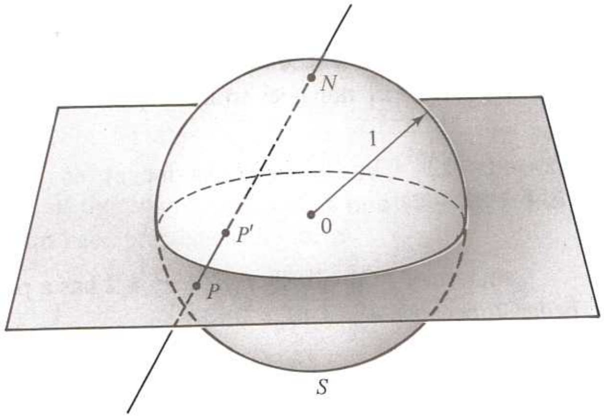

FIGURE 5.5

sequence of points z_n $(n = 1, 2, 3, \ldots)$ approaches ∞ if $|z_n|$ can be made arbitrarily large by taking n large.

Consequently we shall write $f(z_0) = \infty$ when $|f(z)|$ increases without bound as $z \to z_0$† and shall write $f(\infty) = w_0$ when $f(z) \to w_0$ as $z \to \infty$. For example, if

$$f(z) = \frac{2z + 1}{z - 1}, \tag{44}$$

then $f(1) = \infty$ and $f(\infty) = 2$.

Observe that for $f(z) = 2z + 1$, we have $f(\infty) = \infty$.

It is convenient in some applications to carry this notion still further and speak of functions which are "analytic at ∞." The analyticity properties of f at ∞ are classified by first performing the mapping $w = 1/z$, which maps the point at infinity to the origin, and then examining the behavior of the composite function $g(w) \equiv f(1/w)$ at the origin $w = 0$. Thus we say

(*i*) $f(z)$ is analytic at ∞ if $f(1/w)$ is analytic (or has a removable singularity) at $w = 0$,‡

(*ii*) $f(z)$ has a pole of order m at ∞ if $f(1/w)$ has a pole of order m at $w = 0$, and

(*iii*) $f(z)$ has an essential singularity at ∞ if $f(1/w)$ has an essential singularity at $w = 0$.

From Theorem 18, we can interpret these conditions for a function analytic outside some disk as follows:

(*i'*) $f(z)$ is analytic at ∞ if $|f(z)|$ is bounded for sufficiently large $|z|$,

†Technically, $f(z_0) = \infty$ if for any $M > 0$ there is a $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies that $|f(z)| > M$.

‡Some authors also allow the possibility of a removable singularity at ∞, but we feel that nothing is gained by this generality.

(*ii'*) $f(z)$ has a pole at ∞ if $f(z) \to \infty$ as $z \to \infty$, and

(*iii'*) $f(z)$ has an essential singularity at ∞ if $|f(z)|$ neither is bounded for large $|z|$ nor goes to infinity as $z \to \infty$.

EXAMPLE 22 Classify the behavior at ∞ of the functions $z^2 + 2$, $(iz + 1)/(z - 1)$, and $\sin z$.

SOLUTION Obviously $f(z) = z^2 + 2$ has a pole at ∞. The pole is of order 2, because

$$f\left(\frac{1}{w}\right) = \frac{1}{w^2} + 2$$

has a pole of order 2 at $w = 0$.

Since

$$\frac{iz + 1}{z - 1} \to i \quad \text{as} \quad z \to \infty,$$

this function is certainly analytic at ∞.

Finally, $\sin z$ has no limit as $z \to \infty$, even for real z (it oscillates). Hence ∞ must be an essential singularity.† ∎

EXAMPLE 23 Find all the functions f which are analytic everywhere in the extended complex plane.

SOLUTION Since f is analytic at ∞, it is bounded for, say, $|z| > M$. By continuity, f is also bounded for $|z| \le M$. Consequently, f is a bounded entire function. Hence f is constant, by Liouville's Theorem. ∎

EXAMPLE 24 Classify all the functions which are everywhere analytic in the extended complex plane except for a pole at one point.

SOLUTION If $f(z)$ has a pole, say, of order m at some finite point z_0, then the Laurent series for f

$$(45)\quad f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges for all $z \neq z_0$. Moreover, since we are assuming that $z_0 \neq \infty$, the function f must be analytic, and hence bounded, at ∞. From Eq. (45), then, we see that the *entire* function defined by the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$

†Alternatively this can be seen directly from the Laurent expansion for $\sin(1/w)$ about $w = 0$.

is also bounded at ∞; thus it must be constant, i.e., equal to a_0. Therefore, the most general form for such a function is

$$(46) \qquad f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 \, .$$

If the pole occurs at $z = \infty$, then $f(1/w)$ has a pole at the origin and can be expressed in the form

$$(47) \qquad f\left(\frac{1}{w}\right) = \frac{a_{-m}}{w^m} + \frac{a_{-m+1}}{w^{m-1}} + \cdots + \frac{a_{-1}}{w} + \sum_{n=0}^{\infty} a_n w^n.$$

Since $f(z)$ is bounded near $z = 0$, $f(1/w)$ is bounded for large $|w|$, and, as before, we conclude that $a_n = 0$ for $n > 0$. Hence Eq. (47) becomes

$$(48) \qquad f(z) = a_{-m} z^m + a_{-m+1} z^{m-1} + \cdots + a_{-1} z + a_0;$$

i.e., $f(z)$ is a *polynomial* in z.

Equations (46) and (48) categorize the totality of all functions possessing one pole in the extended complex plane. ∎

We note in passing that the theory of *Fuchsian equations* is based upon considerations of singularities in the extended complex plane, and these have been extremely helpful in relating many of the so-called "special functions" which arise in mathematical physics; Ref. [3] discusses this application.

Exercises 5.7

1. Classify the behavior at ∞ for each of the following functions.

(a) e^z (b) $\cosh z$ (c) $\dfrac{z-1}{z+1}$

(d) $\dfrac{z}{z^3 + i}$ (e) $\dfrac{z^3 + i}{z}$ (f) $e^{\sinh z}$

(g) $\dfrac{\sin z}{z^2}$ (h) $\dfrac{1}{\sin z}$ (i) $e^{\tan 1/z}$

2. State Picard's Theorem (Sec. 5.6) for functions with an essential singularity at ∞. Verify for e^z.

3. Prove that if $f(z)$ is analytic at ∞, then it has a series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

converging uniformly outside some disk.