Complex Analysis EXTRA EXERCISES

1. Let $z \in \mathbb{C}$ be such that $\operatorname{Im} z > 0$. Prove that $\operatorname{Im} \frac{1}{z} < 0$. (SS **1.1:** 16)

2. Let $z_1, z_2 \in \mathbb{C}$ be such that $z_1 + z_2$ and $z_1 z_2$ are both negative real numbers. Prove that z_1 and z_2 must both be real numbers. (SS **1.1:** 17)

3. Suppose that we wish to describe the set of points z in the complex plane for which $|z-1| = |z-i|$. (Adapted from SS **1.2:** 10(d))

   (a) Try to determine the set of points by drawing a picture.

   (b) Let $z = x + iy$, compute the moduli and set them equal. This will give you a relationship the describes the points of interest.

4. For each of the following, graph the set of points z in the complex plane which satisfy the given equation or inequality. When it is not obvious what the set of points is, show the computations used to determine it. (Adapted from SS **pg 8:** Example 3(b) and **1.2:** 10)

   (a) $\operatorname{Re} z = 3$ (b) $|z - 2 + i| = 2$ (c) $\operatorname{Im} z < -1$

   (d) $|z + 1 - 2i| \leq 3$ (e) $|z| > 5$ (f) $|z + i| = \operatorname{Im} z - 1$

5. (a) Verbally describe the set of points satisfying $|z - (a + bi)| = r$, where $r > 0$.

   (b) Sketch the set of points satisfying $|z - (a + bi)| < r$ where $a > r > b > 0$. Label all points of interest on the real an imaginary axes.

   (c) Sketch the set of points satisfying $|z - (a + bi)| \geq r$, where $a < 0$ and $|a| < r$, $b > r > 0$. Again, label points of interest on the axes.

6. For each of the following,

   - Sketch the number as a vector in the complex plane.
   - Determine the principal argument ϕ of the number.
   - Give the number in polar form.

   (a) $4i$ (b) $3 - 3i$ (c) $-2\sqrt{3} + 2i$

7. Give $-2.34 - 4.81i$ in polar form, with r and ϕ rounded to the nearest hundredth.

8. Give the number in rectangular form.

   (a) $7e^{\frac{3\pi}{4}i}$ (b) $4e^{-\frac{\pi}{6}i}$ (c) $3e^{\frac{27\pi}{4}i}$

9. Give the argument and principal argument of each, denoting them with the correct notation:

   (a) $-4 + 4i$ (b) -7 (c) $1 - \sqrt{3}i$

10. Write $\arg z_1 \bar{z}_2$ in terms of $\arg z_1$ and $\arg z_2$.

11. In this exercise you will see an easy way to construct the identities for $\sin(\theta + \phi)$, $\cos(\theta + \phi)$, $\sin 2\theta$ and $\cos 2\theta$.

(a) Write $e^{(\theta+\phi)i}$ in rectangular form.

(b) Write $e^{(\theta+\phi)i}$ as the product of two exponentials, put each in rectangular form and "foil" the result. Get the result in $a+bi$ form.

(c) Set the real and imaginary parts equal to each other to obtain the first two identities.

(d) Let $\phi=\theta$ to get the second two identities.

12. Use a similar approach and part of your answer to the previous exercise to find an identity for $\cos 3\theta$.

13. For each of the following, find all values of the given root. Then plot all roots on a circle around the origin in the complex plane.

(a) $(-9)^{\frac{1}{2}}$ (b) $(3i)^{\frac{1}{5}}$ (c) $(1-\sqrt{3}i)^{\frac{1}{4}}$ (d) $1^{\frac{1}{5}}$