Math 411 Assignment 15 W14

6 points w/o #4

(1) #1 rethought!

$$|\cos z| = \left| \frac{\exp(iz) + \exp(-iz)}{2} \right|$$

$$= \left| \frac{e^{ix-y} + e^{-ix+y}}{2} \right|$$

$$\leq \frac{1}{2}\left( |e^{ix-y}| + |e^{-ix+y}| \right)$$

$$= \frac{1}{2}\left( |e^{ix}||e^{-y}| + |e^{-ix}||e^{y}| \right)$$

+1 for anything

$$= \frac{1}{2}\left( |e^{-y}| + |e^{y}| \right) \leq \frac{1}{2}(e^{-y} + e^{y}) \neq \boxed{e^{|y|}}$$

Better: One of these is always $\leq 1$ and the other is always $\leq e^{|y|}$, so

$$|\cos z| \leq \frac{1}{2}(e^{|y|} + 1)$$

$$|\sin z| = \left| \frac{\exp(iz) - \exp(-iz)}{2i} \right|$$

$$= \left| \frac{e^{ix-y} - e^{-ix+y}}{2i} \right|$$

$$\leq \frac{1}{2}\left( |e^{ix-y}| + |e^{ix+y}| \right)$$

$$= \frac{1}{2}\left( |e^{ix}||e^{-y}| + |e^{ix}||e^{y}| \right)$$

$$= \frac{1}{2}(e^{-y} + e^{y})$$

(2) a) $|z^2+5z-2| \leq |z^2|+|5z|+|-2|$

$= |z|^2+5|z|+2$

$+\frac{1}{2}$

$\leq |5i|^2+5|5i|+2 = 25+25+2 = 52$

b) $\left|\dfrac{z^2+5z-2}{z-2i}\right| = \dfrac{|z^2+5z-2|}{|z-2i|} = \dfrac{|z^2+5z-2|}{3} \leq \dfrac{52}{3}$

$+\frac{1}{2}$

c) $\left|\int_\gamma \dfrac{z^2+5z-2}{z-2i}\,dz\right| \leq \max_{z\in\gamma}\left|\dfrac{z^2+5z-2}{z-2i}\right| \cdot \text{length}(\gamma)$

$+\frac{1}{2}$

$= \dfrac{52}{\cancel{3}} \cdot 2\pi(\cancel{3}) = 104\pi$

d) $\int_\gamma \dfrac{z^2+5z-2}{z-2i}\,dz = 2\pi i \cdot \left((2i)^2+5(2i)-2\right)$

$= 2\pi i(-6+10i)$

$+\frac{1}{2}$

$= -20\pi - 12\pi i$

$= 4\sqrt{34}\,\pi$

e) $|-20\pi-12\pi i| = \sqrt{400\pi^2+144\pi^2} = \pi\sqrt{544} < 104\pi$

$23.32\pi$

$+\frac{1}{2}$

Math 411, Assignment 15, continued

(4) (d) $B_1 = \{ z \in \mathbb{C} : -\pi < \operatorname{Arg} z \le -\frac{\pi}{3} \}$

$B_2 = \{ z \in \mathbb{C} : -\frac{\pi}{3} < \operatorname{Arg} z \le \frac{\pi}{3} \}$

$B_3 = \{ z \in \mathbb{C} : \frac{\pi}{3} < \operatorname{Arg} z \le \pi \}$

(e) $(-8i)^{\frac{1}{3}} = 2e^{\left(\frac{\pi}{2} + \frac{2\pi}{3}\right)i} = 2e^{\frac{7\pi}{6} i} = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2} i\right) = -\sqrt{3} - i$

(f) $B_1 = \{ z \in \mathbb{C} : \theta \le \operatorname{Arg} z < \theta + \frac{2\pi}{3} \}$

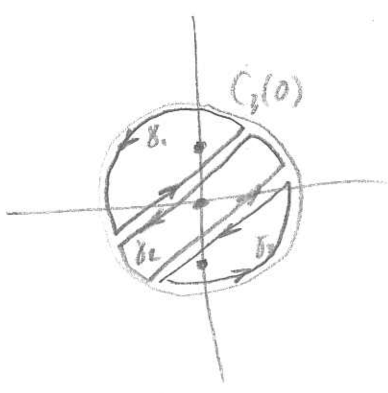
$B_2 = \{ z \in \mathbb{C} : \theta + \frac{2\pi}{3} \le \operatorname{Arg} z < \theta + \frac{4\pi}{3} \}$

$B_3 = \{ z \in \mathbb{C} : \theta + \frac{4\pi}{3} \le \operatorname{Arg} z < \theta + 2\pi \}$

$f(z) = \frac{z^2+1}{z^2+4}$

$f'(z) = \frac{2z(z^2+4) - 2z(z^2+1)}{(z^2+4)^2} = \frac{6z}{(z^2+4)^2}$

(5) $$\int_{C_3(0)} \frac{z^2+1}{z^4+4z^2}\,dz = \int_{C_3(0)} \frac{z^2+1}{z^2(z+2i)(z-2i)}\,dz$$




$$= \int_{\gamma_1} \frac{(z^2+1)/[z^2(z+2i)]}{z-2i}\,dz + \int_{\gamma_2} \frac{(z^2+1)/(z^2+4)}{z^2}\,dz$$

$$+ \int_{\gamma_3} \frac{(z^2+1)/[z^2(z-2i)]}{z+2i}\,dz$$

$$= 2\pi i \cdot \frac{(2i)^2+1}{(2i)^2(2i+2i)} + 2\pi i \cdot \frac{6(0)}{(0^2+4)^2} + 2\pi i \cdot \frac{(-2i)^2+1}{(-2i)^2(-2i-2i)}$$

$$= 2\pi i \cdot \frac{-3}{(-4)(4i)} + 2\pi i \cdot \frac{-3}{(-4)(-4i)}$$

$$= 0$$

Math 411, Assignment 15, continued

w14

(3) (a) $|3z^2-5z| \geq 100$ Want

(b) $|3z^2-5z| \geq \frac{1}{2}|3||z|^2$ for $z \geq R_0$

Know

$\frac{1}{2}|3|R_0^2 = 100$

$R_0^2 = \frac{200}{3}$ +½

$R_0 = \sqrt{\frac{200}{3}}$

(c) $\left|\frac{1}{3z^2-5z}\right| \leq \varepsilon \Longrightarrow |3z^2-5z| \geq \frac{1}{\varepsilon}$

$|3z^2-5z| \geq \frac{1}{2}(3)R_0^2 = \frac{1}{\varepsilon}$ +½

$R_0^2 = \frac{2}{3\varepsilon}$

$R_0 = \sqrt{\frac{2}{3\varepsilon}}$

Assignment 15½/16 #4 only
3 points

(4) (a) $B_1 = \{z \in \mathbb{C} : 0 \leq \operatorname{Arg} z < \frac{2\pi}{3}\}$

$B_2 = \{z \in \mathbb{C} : \frac{2\pi}{3} \leq \operatorname{Arg} z < \frac{4\pi}{3}\}$

$B_3 = \{z \in \mathbb{C} : \frac{4\pi}{3} \leq \operatorname{Arg} z < 2\pi\}$ +½ each

(b) $(-8i)^{\frac{1}{3}} = \left(8e^{\left(\frac{3\pi}{2}+2\pi n\right)i}\right)^{\frac{1}{3}} = 2e^{\left(\frac{\pi}{2}+\frac{2\pi}{3}n\right)i}$

root in 3rd branch is $2e^{\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)i} = 2e^{\frac{11\pi}{6}i} = 2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$

$= \sqrt{3}-i$

(c)