Math 411 ASSIGNMENT 17 **Due at 3 PM Monday, February 17th**

Geometric Series: $1 + z + z^2 + z^3 + z^4 + \cdots = \dfrac{1}{1-z}$ if $|z| < 1$ and diverges otherwise.

Convergence of Power Series: For the power series $\displaystyle\sum_{k=0}^{\infty} c_k(z-z_0)^k$, compute

$$R = \lim_{k\to\infty} \left|\frac{c_k}{c_{k+1}}\right| \qquad \text{or} \qquad R = \frac{1}{\displaystyle\lim_{k\to\infty} |c_k|^{1/k}}$$

If $R = 0$ the series converges only at z_0, and if $R = \infty$, the series converges for all complex numbers. If $0 < R < \infty$,

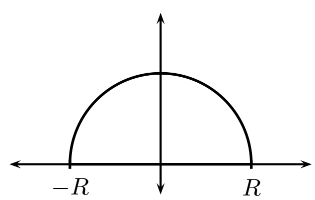
- the series converges for all z with $|z - z_0| < R$ and diverges for all z with $|z - z_0| > R$
- the series may or converge or diverge for each z with $|z - z_0| = R$.

Some Common Power Series:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \qquad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots.$$

Exercises:

1. Do Exercises 24 (a) and (b) on page 87 of 4G. You've done this kind of thing already in this class...
2. Do Exercises 25 (b) and (c) on page 87 of 4G, using the power series given above.
3. Use whichever if the two methods above for determining radius of convergence that is most appropriate to do each of Exercises 30(a),(b),(g) and (e) on page 88 of 4G.
4. For each of Exercise 28(a) and (b) on page 88 of 4G,
   (a) identify the functions f_k of Proposition 7.22
   (b) find a sequence M_k of real numbers such that both $|f_k(z)| \le M_k$ for all z in the region given *AND* $\displaystyle\sum_{k=1}^{\infty} M_k < \infty$
   (c) justify that $\displaystyle\sum_{k=1}^{\infty} M_k < \infty$
5. In this exercise you will be working with the integral of $f(z) = \dfrac{e^{iz}}{z^2+1}$ over the closed contour shown to the right.



   (a) Find the value of the integral when $R > 1$.
   (b) Find a *number* bound for e^{iz} for z in the upper half plane $\operatorname{Im} z \ge 0$.
   (c) Find a bound (involving R) for $\dfrac{e^{iz}}{z^2+1}$ on the semi-circle portion of the contour.
   (d) Find a bound on $\displaystyle\int_\gamma \frac{e^{iz}}{z^2+1}\,dz$, where γ is the semi-circle portion of the contour.
   (e) Based on your answer to (d), what happens to the integral over the semi-circle as $R \to \infty$?
   (f) Because of your answer to (e), we can now conclude that $\displaystyle\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{it}}{t^2+1}\,dt$ has the value you found in part (a). Apply Euler's formula to break this integral into a sum of two other integrals. Pull the i out of the second one.
   (g) What is the value of the integral of an odd real-valued function from $-a$ to a? Is $\sin t$ even or odd? How about $\dfrac{1}{t^2+1}$? How about $\dfrac{\sin t}{t^2+1}$? Conclusion from all this?
   (h) We now know that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\,dt = \cdots$