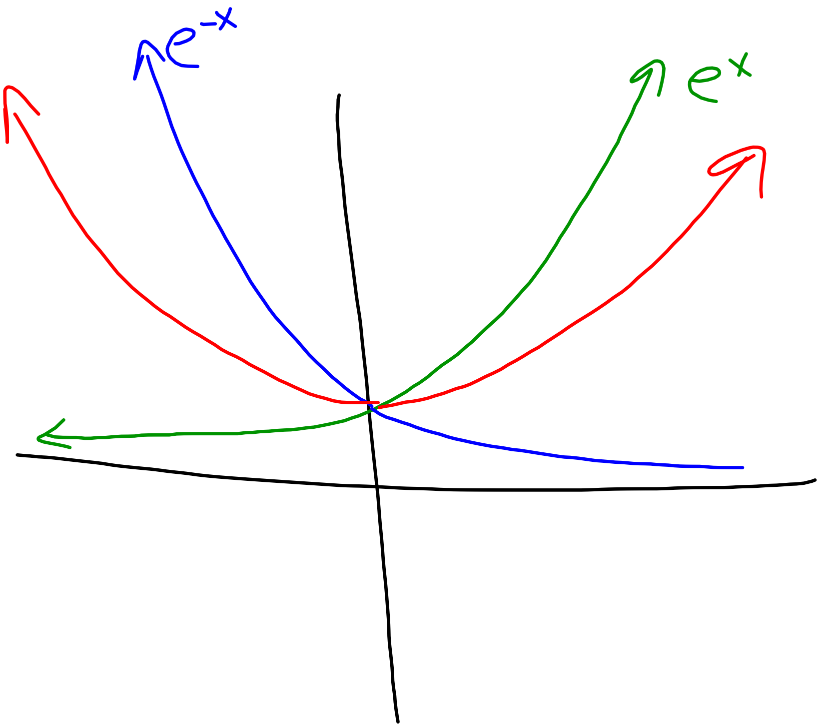
$$\cosh z = \frac{e^{z}+e^{-z}}{2}, \quad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

$$\cosh(-z) = \frac{e^{-z}+e^{-(-z)}}{2} = \frac{e^{-z}+e^{z}}{2} = \cosh(z)$$

$$\sinh(-z) = \frac{e^{-z}-e^{-(-z)}}{2} = \frac{e^{-z}-e^{z}}{2} = -\frac{e^{z}-e^{-z}}{2} = -\sinh(z)$$

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$



# Finding Bounds

Have some $f(z)$ or some $\int_\gamma f(z)\,dz$ and we want to say

$$\left.\begin{aligned} &|f(z)| \le \text{something} \\ \text{or } &\left|\int_\gamma f(z)\,dz\right| \le \text{something} \end{aligned}\right\} \text{Bounding}$$

Sometimes we want

$$|f(z)| \geq \text{something} > 0$$

Bounding away from zero.

# Tools:

- $|z_1+z_2| \le |z_1| + |z_2|$ triangle inequality

- $\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma \underbrace{|f(z)|}_{\le \max_{z\in\gamma}|f(z)|}|dz|$

$$\le \max_{z\in\gamma}|f(z)| \int_\gamma |dz|$$

$$= \max_{z\in\gamma}|f(z)| \cdot \text{length}(\gamma)$$

- $|a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0|$

$$= \left|a_n z^n\left(1 + \frac{a_{n-1}}{a_n z} + \frac{a_{n-2}}{a_n z^2} + \cdots \frac{a_0}{a_n z^n}\right)\right|$$

$$= |a_n||z|^n \underbrace{\left|1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_0}{a_n z^n}\right|}_{\text{can get between } \frac{1}{2} \text{ and } 2}$$

$\to 1$ as $z$ gets big

$$\boxed{\frac{1}{2}|a_n||z|^n \le |a_n z^n + \cdots| \le 2|a_n||z|^n}$$

Bound $2z^2+3z$ on $\gamma=\gamma(\theta)=2+3e^{i\theta}$

$0\le\theta\le\pi$

$$|2z^2+3z| \le |2z^2|+|3z|$$ triangle inequality

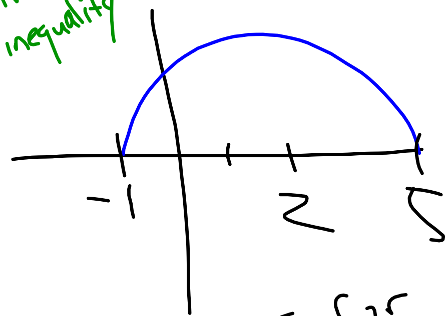
$$= 2|z|^2+3|z|$$

$|z_1z_2| = |z_1||z_2|$

$\boxed{|z^n| = |z|^n}$

$$\le 2(5)^2+3(5)$$

$$= 65$$



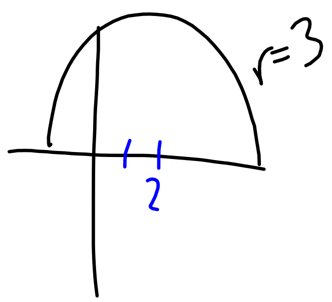

$|z|\le 5$ for all $z\in\gamma$

Bound $\int_\gamma (2z^2+3z)\,dz$ on the same $\gamma$.

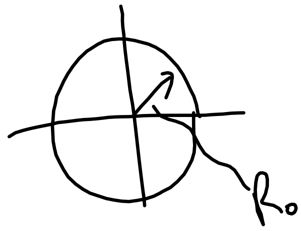
$$\left|\int_\gamma (2z^2+3z)\,dz\right| \le \max_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma)$$

$$= 65\left(\frac{\cancel{2}\pi\cdot 3}{\cancel{2}}\right)$$

$$= 195\pi$$



Find $R_0$ such that




$$\left|\frac{1}{2z^2+3z}\right| \le \frac{1}{10{,}000}$$ for $z$ on

$\gamma(\theta) = R_0 e^{i\theta}$

$$\frac{1}{|2z^2+3z|} \le \frac{1}{10{,}000} \qquad 0 \le \theta \le 2\pi$$

$$|2z^2+3z| \ge 10{,}000$$

Lemma 5.6 says there is an $R_0$ such that $|z| \ge R_0$ gives

$$\underset{p(z)}{|2z^2+3z|} \ge \tfrac{1}{2}(2)|z|^2 = |z|^2 \ge R_0^2$$

$$10{,}000 = R_0^2$$

$$R_0 = 100$$

If $|z| \ge 100$, $\left|\dfrac{1}{2z^2+3z}\right| \le \dfrac{1}{10{,}000}$

$(zi+2)(-zi-2)$

$-z^2-4$

$-(z^2+4)$

$zi+2=0$

$\frac{zi}{i}=-\frac{2}{i}$

$z=2i$

$(z+2i)(z-2i)$

$z=-2i \quad z=2i$

$-zi-2=0$

$-zi=2$

$z=\frac{2}{-i}$

$z=+\frac{2}{i}\cdot\frac{i}{\cancel{i}}$