Classification of Singularities—Definition

removable singularity — Laurent series has no "negative terms"

pole — Laurent series has a finite number of negative terms

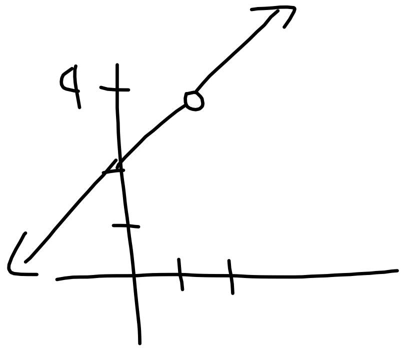
essential singularity — Laurent series has an infinite number of negative terms.

$$f(z) = \frac{z^2-4}{z-2} = \frac{(z+2)(z-2)}{z-2} = z+2$$

$z \neq 2$

$$f(z) = \begin{cases} z+2 & \text{if } z \neq 2 \\ \cancel{\text{undef} \quad \text{if } z = 2} & \\ 4 & \text{when } z = 2 \end{cases}$$

redefined



$$\lim_{z\to 2} f(z) = \lim_{z\to 2} \frac{z^2-4}{z-2} = \lim_{z\to 2} (z+2) = 4$$

(understood that $z \neq 2$)

Laurent series for $f(z) = \frac{z^2-4}{z-2}$
(centered at 2)

$$f(z) = \frac{(z+2)(z-2)}{z-2} = \frac{(z-2+4)(z-2)}{(z-2)}$$

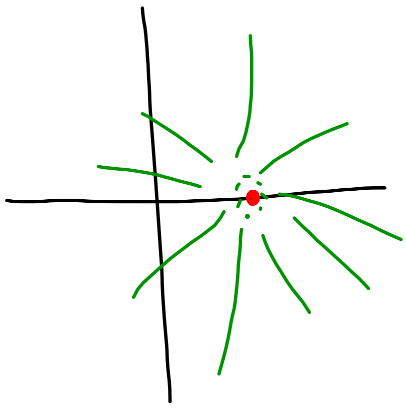
$$\sum_{k=-\infty}^{\infty} c_k (z-2)^k$$

$$= \frac{(z-2)^2 + 4(z-2)}{(z-2)}$$

$$= \frac{(z-2)^2}{z-2} + \frac{4(z-2)}{z-2}$$

$$= 4 + (z-2)$$

$c_0 = 4$
$c_1 = 1$
All others are zero



$$g(z) = e^{\frac{1}{z}} \qquad e^u = 1 + u + \frac{1}{2}u^2 + \frac{1}{3!}u^3 + \cdots$$

$$= 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \cdots$$

$$+ \frac{1}{3!z^3} + \frac{1}{2z^2} + \frac{1}{z} + 1 = e^{\frac{1}{z}}$$

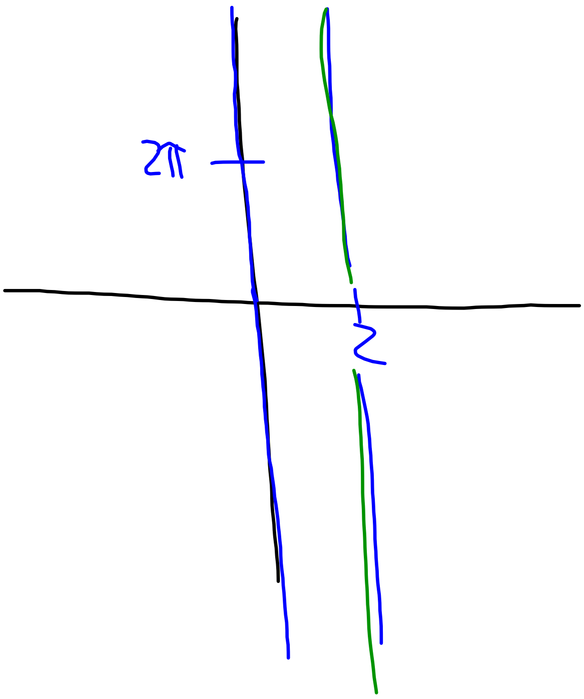
$$+ \frac{1}{2!}z^{-2} + z^{-1} + 1 = e^{\frac{1}{z}}$$

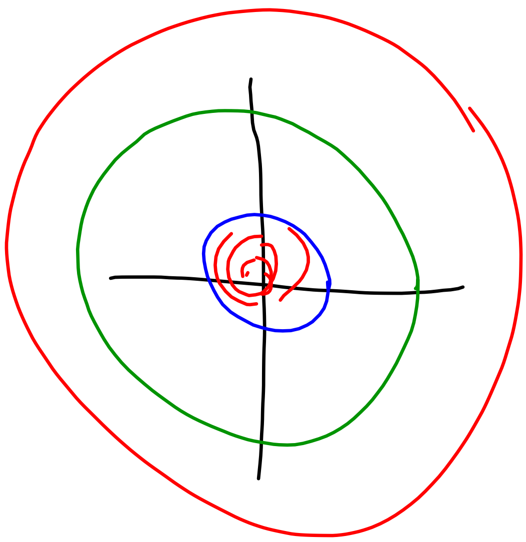
$C_k = \frac{1}{k!}$ for $k = 0, 1, 2, 3, \ldots$

$z = 0$ is an essential singularity of $e^{\frac{1}{z}}$

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

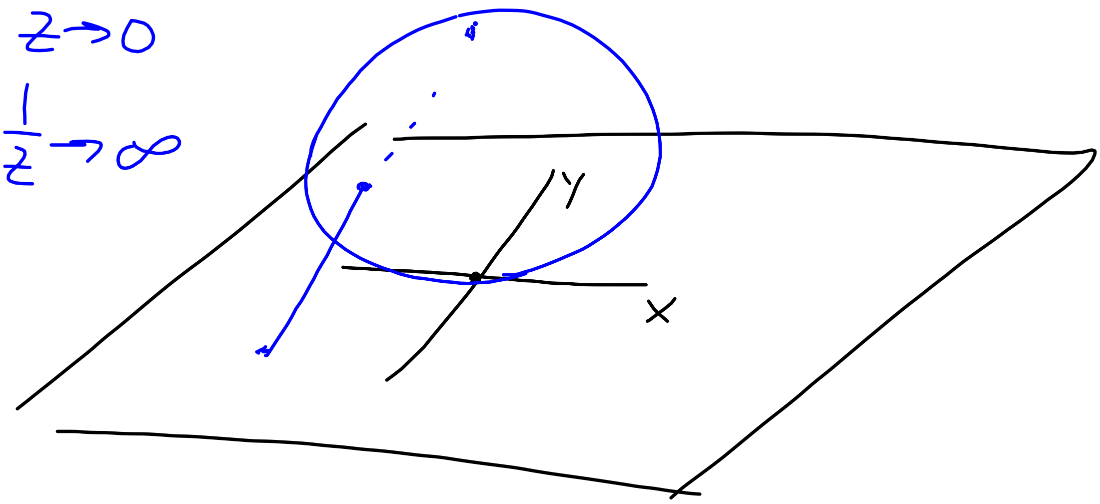




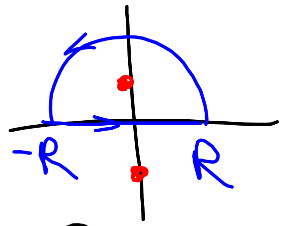
$e^{\frac{1}{z}}$ $\quad$ e^{z}

$z \to 0$

$\frac{1}{z} \to \infty$



Result: $$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\,dt = \frac{\pi}{e}$$



$$\frac{\pi}{e} = \int_{\frown} \frac{e^{iz}}{\underset{(z+i)(z-i)}{z^2+1}}\,dz = \underbrace{\int_{\frown} \frac{e^{iz}}{z^2+1}\,dz}_{\text{Show this goes to zero as } R\to\infty} + \int_{-} \frac{e^{iz}}{z^2+1}\,dz$$

$$= \int \frac{\cos x}{x^2+1}\,dx + i \cancel{\int \frac{\sin x}{x^2+1}\,dx} \to 0$$