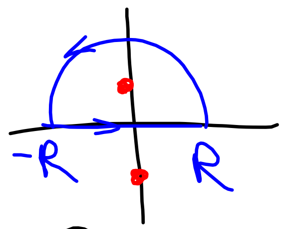
Result:

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\,dt = \frac{\pi}{e}$$



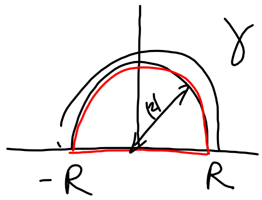
$$\frac{\pi}{e} = \int_{\frown} \frac{e^{iz}}{\underset{(z+i)(z-i)}{z^2+1}}\,dz = \underbrace{\int_{\frown} \frac{e^{iz}}{z^2+1}\,dz}_{\text{Show this goes to zero as } R\to\infty} + \int_{-} \frac{e^{iz}}{z^2+1}\,dz$$

$$= \int \frac{\cos x}{x^2+1}\,dx + i\cancelto{0}{\int \frac{\sin x}{x^2+1}\,dx}$$

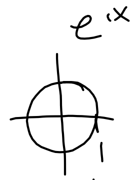
$$\int_\gamma \frac{e^{iz}}{z^2+1}\,dz = \int_\gamma \frac{e^{iz}/(z+i)}{z-i}\,dz$$

$$= 2\pi i \left[\frac{e^{iz_0}}{z_0+i}\right]_{z_0=i}$$

$$= \pi/e$$

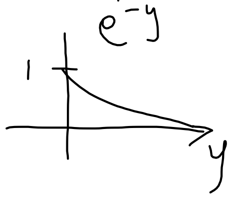


b) $$|e^{iz}| = |e^{i(x+iy)}|$$



$$= |e^{-y+ix}|$$

$$= |e^{-y}||e^{ix}|$$



$$\leq (1)(1) = 1$$

b) $$\left|\frac{e^{iz}}{z^2+1}\right| = |e^{iz}|\left|\frac{1}{z^2+1}\right|$$

$$\frac{1}{2}|z|^2 \leq |z^2+1| \quad > |z^2+1|$$

$$\frac{1}{\frac{1}{2}|z|^2} \geq \frac{1}{|z^2+1|}$$

So $$\left|\frac{e^{iz}}{z^2+1}\right| \leq \frac{1}{\frac{1}{2}|z|^2} = \frac{2}{|z|^2}$$

$$= \frac{2}{R^2}$$

$$\left|\int_\gamma \frac{e^{iz}}{z^2+1}\,dz\right| = \max|M| \cdot \text{length}$$

$$= \frac{2}{R^2} \cdot \pi R$$

$$= \frac{2\pi}{R}$$

$$\lim_{R\to\infty} \frac{2\pi}{R} = 0$$

Lemma A.7.5: Suppose p and q are analytic at z_0, $p(z_0) \neq 0$ and q has a zero of order m at z_0. Then $\frac{p}{q}$ has a pole of order m at z_0.

WTS $\frac{p}{q}(z) = \frac{g(z)}{(z-z_0)^m}$

Proof: $q(z) = (z-z_0)^m g(z)$ for some g that is analytic at z_0 and such that $g(z_0) \neq 0$.

Let $h(z) = \frac{p(z)}{g(z)}$. Then h is analytic at z_0 and $h(z_0) \neq 0$.

$$\frac{h(z)}{(z-z_0)^m} = \frac{p(z)}{q(z)}$$

$$g(z) = (z-z_0)^m \frac{p(z)}{h(z)} = q$$

Now $g(z) = \frac{p(z)}{h(z)}$, giving

$$q(z) = (z-z_0)^m \frac{p(z)}{h(z)}.$$

Therefore

$$\frac{p(z)}{q(z)} = \frac{h(z)}{(z-z_0)^m},$$

so z_0 is a pole of $\frac{p}{q}$, of order m.

Where is $\sin z = 0$?

$$0 = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

When is $0 = e^{iz} - e^{-iz}$?

$$= e^{i(x+iy)} - e^{-i(x+iy)}$$

$$= e^{-y+ix} - e^{y-ix}$$

$$= e^{-y}e^{ix} - e^{y}e^{-ix}$$

$e^{-y} = e^{y}$

$e^{y}e^{-y} = e^{y}e^{y}$

$1 = e^{2y}$

$2y = 0$

$y = 0 \implies z = x$ $\quad \sin x = 0$ when $x = n\pi$