$$g(z) = \frac{1}{e^z - 1} = \frac{1}{f}$$ Lemma A.8

What order are zeros of f?

Where are singularities?

Where $e^z - 1 = 0$, so $z = 0$ for sure

$$\boxed{e^z = 1} = re^{i\theta}$$

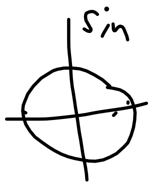
$$e^{x+iy} = e^x(\cos y + i \sin y) = 1 + 0i$$

pos

$e^x \underbrace{\cos y}_{\substack{\pm 1 \\ @\pi n}} = 1$ $\qquad$ $e^x \sin y = 0$

$\sin y = 0$

$y = \pi n$



Can't be -1, so

$y = 2\pi n$

At $y = 2\pi n$, $\cos y = 1 \Rightarrow e^x = 1 \Rightarrow x = 0$

$$\boxed{z = 0 + 2\pi n i}$$ zeros of $e^z - 1$

singularities of $\frac{1}{e^z - 1}$

$f(z) = e^z - 1$, zeros at $z = 2\pi n i$

$f(2\pi n i) = 0$

$f'(z) = e^z$

$f'(2\pi n i) = 1 \neq 0$

poles of order 1

$$g(z) = \frac{1}{e^z - 1}$$

$f(z) = e^z - 1$ has zeros $z = 2\pi n i$.

That is, $f(2\pi n i) = 0$.

$f'(z) = e^z$ and $f'(2\pi n i) = 1 \neq 0$, so by def A.1

$z = 2\pi n i$ are zeros of order 1 of f.

Therefore g has poles of order 1 at $z = 2\pi n i$, by Lemma A.8.

Function $u(x,y)$ is <u>harmonic</u> if it satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \frac{\partial u}{\partial t}$$

(Laplace's equation, steady-state heat equation.)

Show that $u(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ is harmonic.

$(\tan^{-1}x)' = \frac{1}{1+x^2}$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

Aside:

$\frac{y}{x} = yx^{-1}$

$\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -yx^{-2} = -\frac{y}{x^2}$

$$= \frac{-1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{y}{x^2}$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$\frac{y}{x} = \frac{1}{x} y$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{x}{x^2}$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y}=\frac{2xy}{(x^2+y^2)^2}+\frac{-2xy}{(x^2+y^2)^2}=0$$