$u(x,y)$ is harmonic if it is a solution to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

real-valued

$f(z) = f(x,y) = u(x,y) + iv(x,y)$

complex valued

If f is holomorphic, then both u and v are harmonic.

C.3 Give u, v harmonic that satisfy

CR Eqns. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

make

$f(z) = f(x,y) = u(x,y) + iv(x,y)$

$u(x,y) = x^3 - 3xy^2 + y$

Find the harmonic conjugate v of u.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$$

So, $v(x,y) = 3x^2y - y^3 + \phi(x)$

$$-\frac{\partial v}{\partial x} = -\left(6xy + \phi'(x)\right)$$

$$= -6xy - \phi'(x) = \frac{\partial u}{\partial y} = -6xy + 1$$

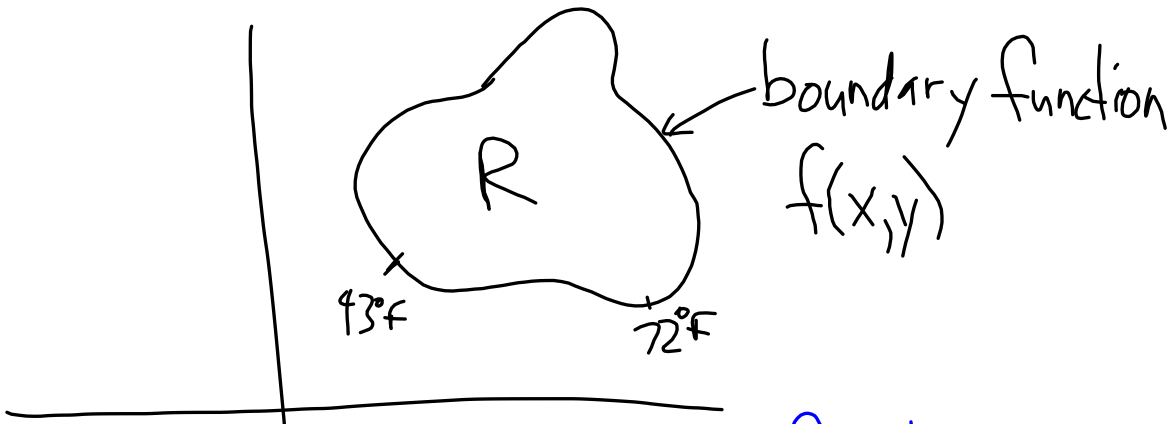
$$-\phi'(x) = 1$$

$$\phi'(x) = -1$$

$$\phi(x) = -x + C$$

$v(x,y) = 3x^2y - y^3 - x + C$

Diri~~chlet~~elict Problem:



Goal: Find a function $u(x,y)$ that is harmonic in R and such that $u(x,y) = f(x,y)$ on the boundary.

Laplace transform

$$f(t) \xmapsto{\text{Laplace transform}} F(s)$$

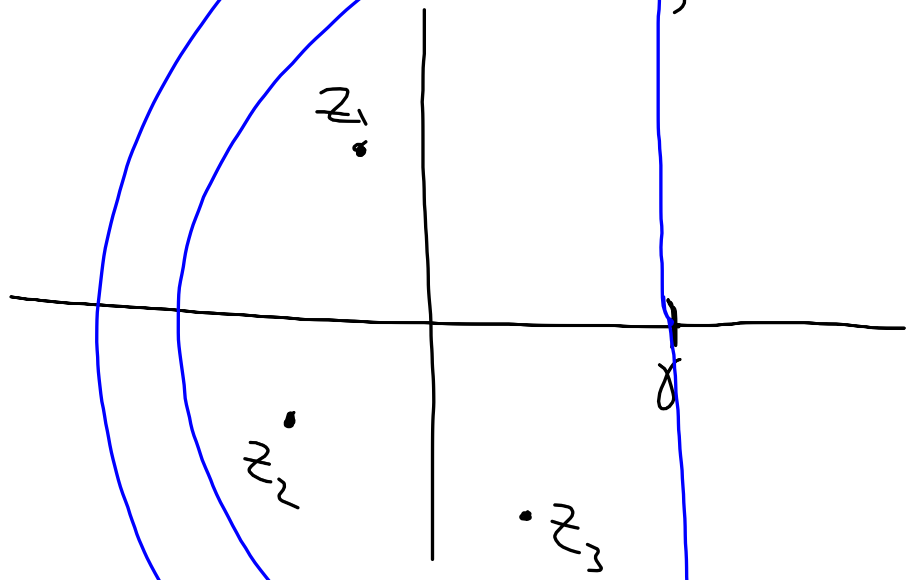
$$\xleftarrow{}\!\!\!| \quad \text{inverse Laplace transform}$$

$f(t)$	$F(s)$

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad s>0$$

Have $\underline{F(s)}$, want $f(t)$

$\to$ singularities at z_1, z_2, z_3



$$f(t) = PV\int_{-\infty}^{\infty} F(s)e^{st}\,ds$$

$$= \lim_{R\to\infty} \int_{\gamma - iR}^{\gamma + iR} F(s)e^{st}\,ds$$

$$2\pi i \sum \mathrm{Res} = \int \bigcirc$$

$$= \underbrace{\int \bigcirc}_{\to \text{ inverse LT as } R \to \infty} + \underbrace{\int \bigcirc}_{\to 0 \text{ as } R \to \infty}$$

Assn. 23

Verify each is harmonic, then find its harmonic conjugate:

1. $u(x,y) = xy - x + y$
2. $u(x,y) = e^x \sin y$