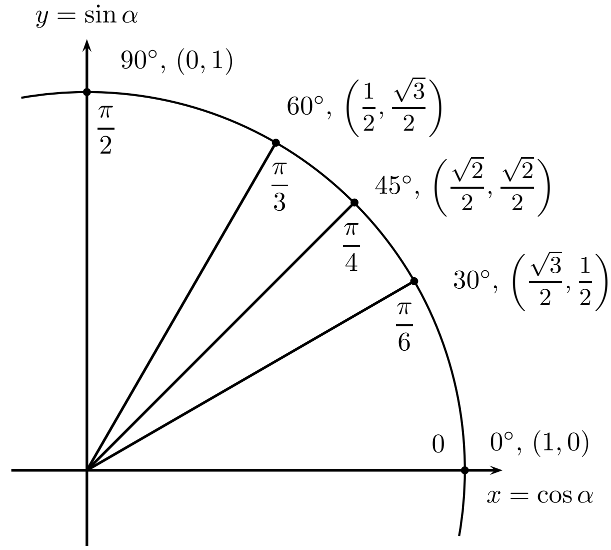
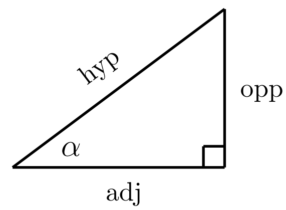
Trigonometric Functions of Special Angles

α, degrees	α, radians	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	undefined



Trigonometric Functions of an Acute Angle of a Right Triangle

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\alpha = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\alpha = \frac{\text{opp}}{\text{adj}},$$



Pythagorean Identity

$$\sin^2\alpha + \cos^2\alpha = 1$$

Sum and Difference Identities

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Double Angle Identities

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Complex Numbers

A **complex number** z is a number of the form $z = x + iy$, where x and y are real numbers and $i^2 = -1$. The numbers x and y are called the **real part** and **imaginary part** of z, denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$.

Complex Conjugates

The **complex conjugate** (or just conjugate) of a number $z = x + iy$ is the number $\bar{z} = x - iy$. The following hold:

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2 \qquad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \qquad \operatorname{Re} z = \frac{z + \bar{z}}{2} \qquad \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

Modulus of a Complex Number

The **modulus** (or **absolute value**) of a number $z = x + iy$ is the *real* number $|z| = \sqrt{x^2 + y^2}$. The following hold:

$$|z|^2 = z\bar{z} \qquad |z_1 z_2| = |z_1||z_2| \qquad |\bar{z}| = |z| \qquad |z_1 + z_2| \leq |z_1| + |z_2|$$

$$|\pm z_1 \pm z_2| \leq |z_1| + |z_2| \qquad |\pm z_1 \pm z_2| \geq ||z_1| - |z_2|| \qquad \left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$$

Argument of a Complex Number

For any complex number $z = x + iy$, there exist real numbers $r \geq 0$ and phi such that

$$z = x + iy = r(cos\phi + i \sin \phi).$$

This is sometimes called the **polar form** of z. The radian value ϕ is an **argument** of z, denoted by $\phi = \arg z$. Note the following:

- If ϕ is an argument of z, then so is $\phi + 2\pi n$ for any *integer* n.
- The value of ϕ (there is only one) such that $-\pi < \phi \leq \pi$ is called the **principal argument** of z, denoted by $\operatorname{Arg} z$.
- $r = |z|$

Euler's Formula and Exponential Form

- For any real number ϕ we *define* $e^{i\phi} = \cos\phi + i \sin\phi$. This is called **Euler's formula**.
- Using Euler's formula, for any complex number z there exists a real number ϕ such that
$$z = r(\cos\phi + i\sin\phi) = re^{i\phi}.$$
$re^{i\phi}$ is called the **exponential form** of z.

Topology of the Complex Plane

In all of the following and throughout our course, it is understood that a set S under discussion is a subset of the complex plane.

- For $z_0 \in \mathbb{C}$ and $\varepsilon > 0$ the set $D_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon$ is called an ε **neighborhood** of z_0.
- A point z_0 is an **interior point** of S if there is an ε neighborhood of z_0 that contains only points of S.
- A point z_0 is an **exterior point** of S if there is an ε neighborhood of z_0 that contains no points of S.
- A point z_0 is a **boundary point** of S if every ε neighborhood of z_0 contains points of S and points not in S.
- A set S is **open** if every point of S is an interior point.
- A set S is **closed** if S contains all of its boundary points.
- The set of all boundary points of S is called the **boundary** of S, denoted by ∂S.
- The **closure** of a set S is the union of S with all its boundary. The closure of S is denoted $\overline{S}$ or ∂S.
- An open set S is **connected** if any two points of S can be joined by a polygonal line segment.
- A point z_0 is called an **accumulation point** of a set S if every ε neighborhood of z_0 contains some point in S other than z_0.
- A point $z_0 \in S$ is called an **isolated point** if there exists an ε neighborhood of z_0 containing no points of S other than z_0.

Deleted Neighborhood

A **deleted neighborhood** of a point z_0 is a set of the form

$$\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$$

for some $\varepsilon > 0$. Given this, an accumulation point z_0 of a set S is a point such that every deleted neighborhood of z_0 contains at least one point in S.

Function

A **function** f is a rule that assigns to each $z \in G$, where G is some subset of $\mathbb{C}$, a unique complex number w. We indicate this by writing $w = f(z)$.

- We sometimes say that z is **mapped to** w by f, and that f is a **mapping** from the complex numbers to the complex numbers.
- The number w is called the **image** of z.
- The set G is called the **domain** of f, and the **range** of f is the set

$$\{w \in \mathbb{C} : w = f(z) \text{ for some } z \in G\}.$$

Definition 2.1: **Limit of a Function (Churchill and Brown)**

Suppose a function f is defined at all points in a deleted neighborhood of a point z_0. We say that the limit of f as z approaches z_0 is a number w_0 if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta.$$

We write $\lim_{z \to z_0} f(z) = w_0$.

Definition 2.1: **Limit of a Function (Beck, Marchesi, Pixton, Sabalka)**

Suppose f is a complex function with domain G and z_0 is an accumulation point of G. Suppose there is a complex number w_0 such that for every $\varepsilon > 0$, we can find a $\delta > 0$ so that for all $z \in G$ satisfying $0 < |z - z_0| < \delta$ we have $|f(z) - w_0| < \varepsilon$. Then w_0 is **the limit of** f **as** z **approaches** z_0, and we write $\lim_{z \to z_0} f(z) = w_0$.

Lemma 2.4: **Limit Rules**

Let f and g be complex functions and let c and z_0 be complex numbers. If $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ exist, then

(a$_1$) $\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$

(a$_2$) $\lim_{z \to z_0} [c\, g(z)] = c \lim_{z \to z_0} g(z)$

(b) $\lim_{z \to z_0} [f(z) \cdot g(z)] = \left[\lim_{z \to z_0} f(z)\right] \cdot \left[\lim_{z \to z_0} g(z)\right]$

(c) If $\lim_{z \to z_0} g(z) \neq 0$, then $\lim_{z \to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$

Definition 2.7: **Derivative of a Function**

Suppose $f : G \to \mathbb{C}$ is a complex function and z_0 is an interior point of G. The **derivative of** f **at** z_0 is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

when the limit exists. In this case we say that f is differentiable at z_0.

Holomorphic and Entire Functions

We say that f is **holomorphic at** z_0 if it is differentiable at all points in some open disk centered at z_0, and f is holomorphic on an open set G if it is differentiable at all points in G. A function that is differentiable at all points in $\mathbb{C}$ is called **entire**.

Lemma 2.11: Derivative Rules

Suppose that f and g are differentiable at $z \in \mathbb{C}$, $c \in \mathbb{C}$ and $n \in \mathbb{Z}$ and h is differentiable at $g(z)$. Then

(a) $[cf(z)]' = c\,f'(z)$ and $[f(z) + g(z)]' = f'(z) + g'(z)$ (linearity)

(b) $[f(z) \cdot g(z)]' = f(z)g'(z) + g(z)f'(z)$ (product rule)

(c) $\left[\dfrac{f(z)}{g(z)}\right]' = \dfrac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$ (quotient rule)

(d) $(c)' = 0$, $(z)' = 1$ and $(z^n)' = nz^{n-1}$ (power rule)

(e) $[h(g(z))]' = h'(g(z)) \cdot g'(z)$ (chain rule)

Theorem 2.15: Cauchy-Riemann Equations

(a) Suppose that $f(z) = f(x+iy) = u(x,y)+iv(x,y)$ is differentiable at $z_0 = x_0+iy_0$. Then

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(u_0, y_0) \tag{1}$$

(b) Suppose that $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is a complex valued function such that the first partial derivatives of u and v with respect to both x and y exist in an open disk centered at z_0 and are continuous at z_0. If (1) holds at z_0, then f is differentiable at z_0.

In both cases the derivative is given by $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$

Definition 3.15: The Complex Exponential Function

The complex exponential function e^z or $\exp(z)$ is defined for $z = x + iy$ by

$$\exp(z) = e^x(\cos y + i \sin y) = e^x e^{iy}$$

Lemma 3.16: Properties of the Complex Exponential Function

For all $z, z_1, z_2 \in \mathbb{C}$,

(a) $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$

(b) $\exp(-z) = \dfrac{1}{\exp(z)}$

(c) $\exp(z + 2\pi i) = \exp(z)$

(d) $|\exp(z)| = \exp(\operatorname{Re} z)$

(e) $\exp(z) \neq 0$

(f) $\dfrac{d}{dz}\exp(z) = \exp(z)$

Definition 3.17: **The Complex Sine and Cosine Functions**

The complex sine and cosine functions are defined for all $z \in \mathbb{C}$ by

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i} \qquad \text{and} \qquad \cos z = \frac{\exp(iz) + \exp(-iz)}{2}$$

Definition 3.20(a): **The Multiple-Valued Complex Logarithm Function**

For the complex number $z \neq 0$ with any particular argument θ we define

$$\log z = \ln|z| + i(\theta + 2n\pi), \qquad n \in \mathbb{Z}.$$

It is then the case that $e^{\log z} = z$ and $\log e^z = z + 2n\pi i, \ \ n \in \mathbb{Z}$.

Definition 3.20(b): **The Principal Value Complex Logarithm Function**

For any complex number $z \in D^* = \{z \,:\, z \neq re^{i\pi},\ r \geq 0\}$ we define

$$\text{Log}z = \ln|z| + i\text{Arg}z.$$

Then $e^{\log z} = z$, $\text{Log}e^z = z$ and $\dfrac{d}{dz}\text{Log}z = \dfrac{1}{z}$ for all $z \in D^*$.

Definition 1.14: **Paths, Arcs, Contours**

A **path** (or curve, or arc) in $\mathbb{C}$ is the image of a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $[a, b]$ is a closed interval in $\mathbb{R}$. The function γ is the **parametrization** of the path. We also have the following:

- The path (arc) is called **smooth** if it is differentiable.
- The path is called a **closed curve** If $\gamma(a) = \gamma(b)$. It is a **simple closed curve** if $\gamma(s) = \gamma(t)$ only for $s = a$ and $t = b$.
- A path is called **piecewise smooth** if there exist $c_1 < c_2 < \cdots ... < c_n$ in the interval (a, b) such that γ is smooth on each of the intervals $[a, c_1], [c_1, c_2], ..., [c_n, b]$. A piecewise smooth curve is sometimes called a **contour**.

Definition 4.1: **Complex Integration**

Suppose γ is a smooth curve (arc) parameterized by $\gamma(t),\ a \leq t \leq b$, and f is a complex function that is continuous on γ. Then we define the integral of f on γ by

$$\int_\gamma f = \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

If $\gamma(t),\ a \leq t \leq b$ is a piecewise smooth curve (contour) that is differentiable on the intervals $[a, c_1], [c_1, c_2], [c_2, c_3], ..., [c_{n-1}, c_n], [c_n, b]$, then

$$\int_\gamma f = \int_a^{c_1} f(\gamma(t))\gamma'(t)\,dt + \int_{c_1}^{c_2} f(\gamma(t))\gamma'(t)\,dt + \cdots + \int_{c_n}^{b} f(\gamma(t))\gamma'(t)\,dt.$$

Proposition 4.3: **Independence of Parametrization**

Suppose γ is a smooth curve parameterized in the same direction by both

$$\gamma_1(t),\ a \le t \le b \qquad \text{and} \qquad \gamma_2(s),\ c \le s \le ds$$

If f is a complex function that is continuous on γ then

$$\int_\gamma f = \int_a^b f(\gamma_1(t))\gamma_1'(t)\,dt = \int_c^d f(\gamma_2(s))\gamma_2'(s)\,ds$$

Definition 4.4: **Length of a Curve**

The length of a smooth curve γ is

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)|dt$$

for any parametrization $\gamma(t),\ a \le t \le b$ of γ.

Proposition 4.7: **Properties of the Integral**

Suppose that γ is a smooth curve, f and g are complex functions that are continuous on γ, and $c \in \mathbb{C}$.

(a) $\displaystyle\int_\gamma cf = c\int_\gamma f$ and $\displaystyle\int_\gamma (f+g) = \int_\gamma f + \int_\gamma g$

(b) If γ is parameterized by $\gamma(t),\ a \le t \le b$, define the curve $-\gamma$ by $-\gamma(t) = \gamma(a+b-t),\ a \le t \le b$. Then $\displaystyle\int_{-\gamma} f = -\int_\gamma f$.

(c) If γ_1 and γ_2 are curves so that γ_2 starts where γ_1 ends, define $\gamma_1\gamma_2$ by following γ_1 to its end, then continuing on γ_2 to its end. Then $\displaystyle\int_{\gamma_1\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$.

(d) $\displaystyle\left|\int_\gamma f\right| \le \max_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma)$

G-Homotopic Curves

Let $G \subseteq \mathbb{C}$ be open and suppose that γ_0 and γ_1 are curves in G parameterized by

$$\gamma_0(t),\ 0 \le t \le 1 \qquad \text{and} \qquad \gamma_1(t),\ 0 \le t \le 1.$$

We say that γ_0 is **G-homotopic** to γ_1 if there exists a continuous function $h : [0,1] \times [0,1] \to G$ such that $h(t,0) = \gamma_0(t)$ and $h(t,1) = \gamma_1(t)$ for all $t \in [0,1]$. We denote this by $\gamma_0 \sim_G \gamma_1$, and the function h is called a **homotopy**.

THEOREM 4.9: **Cauchy's Theorem**

Suppose that $G \subseteq \mathbb{C}$ is open, f is holomorphic in G. If γ_0 and γ_1 are closed curves in G with $\gamma_0 \sim_G \gamma_1$ via a homotopy h with continuous second partial derivatives, then

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

***G*-Contractible Curve**

Let $G \subseteq \mathbb{C}$ be open. A closed curve γ is said to be ***G*-contractible** if γ is G-homotopic to a point. This is denoted by $\gamma \sim_G 0$.

COROLLARY 4.10:

Suppose that $G \subseteq \mathbb{C}$ is open, f is holomorphic in G, and $\gamma \sim_G 0$ via a homotopy with continuous second partial derivatives. Then

$$\int_{\gamma} f = 0.$$

COROLLARY 4.11:

If f is entire and γ is any smooth closed curve, then

$$\int_{\gamma} f = 0.$$

ANOTHER COROLLARY: **Independence of Path**

Suppose that $G \subseteq \mathbb{C}$ is open and f is holomorphic in G. If γ_0 and γ_1 are curves in G parameterized by

$$\gamma_0(t),\ 0 \le t \le 1 \qquad \text{and} \qquad \gamma_1(t),\ 0 \le t \le 1$$

with $\gamma_0 \sim_G \gamma_1$, $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$, then

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Positively Oriented Curve

A simple closed curve γ is **positively oriented** if it is parameterized so that the inside of the curve is on the left of γ.

Theorem 4.12: **Cauchy's Integral Formula for a Circle**

Let $C_r(z_0)$ be the counterclockwise oriented circle centered at z_0 and with radius r. If f is holomorphic at each point of the closed disk bounded by $C_r(z_0)$, then

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z - z_0}\, dz.$$

Theorem 4.13: **Cauchy's Integral Formula**

Suppose f is holomorphic on the region G, $z_0 \in G$ and γ is a positively oriented, simple closed curve that is G-contractible and such that z_0 is enclosed by γ. Then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0}\, dz.$$

Corollary 4.14:

Suppose f is holomorphic on and inside the circle $C_r(z_0)$ parameterized by $\gamma(\theta) = z_0 + re^{i\theta}$ for $0 \le \theta \le 2\pi$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

Theorem 5.1:

Suppose f is holomorphic on the region G, $z_0 \in G$ and γ is a positively oriented, simple closed curve that is G-contractible and such that z_0 is enclosed by γ. Then

$$f'(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^2}\, dz \qquad \text{and} \qquad f''(z_0) = \frac{1}{\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^3}\, dz$$

Corollary 5.2:

If f is differentiable in the region G, then f is infinitely differentiable in G.

Lemma 5.6:

Suppose $p(z)$ is a polynomial of degree n with leading coefficient a_n. Then there is a real number R_0 such that

$$\tfrac{1}{2}|a_n||z|^n \le |p(z)| \le 2|a_n||z|^n$$

for all z with $|z| \ge R_0$.

THEOREM 5.7: **Fundamental Theorem of Algebra**

Every non-constant polynomial has a root in $\mathbb{C}$.

COROLLARY 5.9: **Liouville's Theorem**

Every bounded entire function is constant.

DEFINITION 5.11: **Antiderivative of a Function**

Let $G \subset \mathbb{C}$ be open and connected. For any functions $f, F : G \to \mathbb{C}$, if F is holomorphic on G and $F'(z) = f(z)$ for all $z \in G$, then F is an **antiderivative of** F on G. An antiderivative of f on G will also sometimes be referred to as a **primitive** of f on G.

THEOREM 5.13: **Second Fundamental Theorem of Calculus**

Suppose that $G \subset \mathbb{C}$ is open and connected, and F is an antiderivative of f on G. If $\gamma \in G$ is a smooth curve with parametrization $\gamma(t)$, $a \leq t \leq b$, then

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a))$$

COROLLARY 5.14: **Independence of Path**

If f is holomorphic on a simply connected open set G, then $\int_\gamma f$ is independent of of the path $\gamma \in G$ between $z_1 = \gamma(a)$ and $z_2 = \gamma(b)$.

COROLLARY 5.15:

Suppose $G \subseteq \mathbb{C}$ is open, γ is a smooth closed curve in G, and f has an antiderivative on G. Then $\int_\gamma f = 0$.

THEOREM 5.17: **First Fundamental Theorem of Calculus**

Suppose $G \subseteq \mathbb{C}$ is a connected open set, and fix some basepoint $z_0 \in G$. For each $z \in G$, let γ_z denote a smooth curve in G from z_0 to z. Let $f : G \to \mathbb{C}$ be a holomorphic function such that, for any simple closed curve $\gamma \in G$, $\int_\gamma f = 0$. Then the function $F : G \to \mathbb{C}$ defined at any point $z \in G$ by

$$F(z) := \int_{\gamma_z} f$$

is holomorphic in G with $F'(z) = f(z)$. (The notation $:=$ means "defined by.")

Definition 5.18: Simply Connected

A connected, open $G \subseteq \mathbb{C}$ is called **simply connected** if every simple closed curve in G is G-contractible. That is, for any simple closed curve $\gamma \subseteq G$, the interior of γ is completely contained in G.

Corollary 5.19

Every holomorphic function on a simply connected open set has a primitive.

Corollary 5.20: Morera's Theorem

Suppose f is continuous in a connected open set G and

$$\int_\gamma f = 0$$

for all smooth closed paths $\gamma \subseteq G$. Then f is holomorphic in G.

Definition 7.1: Convergent and Divergent Sequences

Suppose (a_n) is a sequence of complex numbers. If $a \in \mathbb{C}$ is such that for all $\varepsilon > 0$ there is an integer N for which $|a_n - a| < \varepsilon$ whenever $n \geq N$, then we say the sequence (a_n) is **convergent** with limit a. We write

$$\lim_{n\to\infty} a_n = a.$$

If no such a exists, then we say that (a_n) is **divergent**.

Lemma 7.4

Let (a_n) and (b_n) be convergent sequences and let c be any complex number. Then

(a) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ and $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$

(b) $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

(c) If $\lim b_n \neq 0$ and $b_n \neq 0$ for any n, then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

(d) If f is continuous at a, every a_n is in the domain of f and $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} f(a_n) = f(a)$.

Definition: Monotone Sequence

A sequence (x_n) of real numbers is **non-decreasing** if $x_{k+1} \geq x_k$ for all $k = 1, 2, 3, \ldots$, or **non-increasing** if $x_{k+1} \leq x_k$ for $k = 1, 2, 3, \ldots$. A sequence that is either non-decreasing or non-increasing is called a **monotone** sequence.

Axiom: **Monotone Sequence Property**

Any bounded monotone sequence converges.

Theorem 7.6: **Archimedean Property**

If x is any real number, then there is an integer N that is greater than x.

Lemma 7.7

Suppose $a, b \in \mathbb{C}$ and p is a complex polynomial. Then

- $\lim_{n\to\infty} \frac{p(n)}{a^n} = 0$
- $\lim_{n\to\infty} \frac{b^n}{n!} = 0$

Lemma 7.9

If b_k are nonnegative real numbers then $\sum_{k=1}^{\infty} b_k$ converges if and only if the partial sums are bounded.

Lemma 7.10

If $\sum_{k=1}^{\infty} b_k$ converges, then $\lim_{k\to\infty} b_k = 0$.

Lemma 7.11: (**Test for Divergence**)

If $\lim_{k\to\infty} b_k \neq 0$, then $\sum_{k=1}^{\infty} b_k$ diverges.

Absolute Convergence

A series $\sum_{k=1}^{\infty} c_k$ is **absolutely convergent** if $\sum_{k=1}^{\infty} |c_k|$ converges.

Theorem 7.13

If a series converges absolutely, then it converges.

Lemma 7.16: p**-series**

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Convergence of Sequences of Functions

Let (f_n) be a sequence of functions defined on a set G. If, for each $z \in G$, the sequence $(f_n(z))$ converges, then we say that the sequence (f_n) **converges pointwise** on G. Suppose that (f_n) and f are functions defined on G. If for any $\varepsilon > 0$ there is an N such that for all $z \in G$ and $n \geq N$ we have

$$|f_n(z) - f(z)| < \varepsilon,$$

then (f_n) **converges uniformly** to f on G.

PROPOSITION 7.18

Suppose (f_n) is a sequence of functions that converges uniformly to a function f on a set G. Then f is continuous on G.

PROPOSITION 7.19

Suppose f_n are continuous on the smooth curve γ and converge uniformly on γ to a function f. Then

$$\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f.$$

LEMMA 7.20

If (f_n) is a sequence of functions and M_n is a sequence of constants so that M_n converges to zero and $|f_n(z)| \leq M_n$ for all $z \in G$, then (f_n) converges uniformly to zero on G.

LEMMA 7.21

If (f_n) is a sequence of functions that converges uniformly to zero on G and z_n is any sequence in G, then the sequence $(f_n(z_n))$ converges to zero.

PROPOSITION 7.22

Suppose (f_k) are continuous on the region G, $|f_k(z)| \leq M_k$ for all $z \in G$, and $\sum_{k=1}^{\infty} M_k$ converges. Then $\sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly in G.

DEFINITION 7.24: **Power Series**

A **power series centered at** z_0 is a series of functions of the form $\sum_{k=0}^{\infty} c_k (z - z_0)^k$.

Lemma 7.25: **Geometric Power Series**

The series $\sum_{k=0}^{\infty} z^k$ (which is a power series centered at zero, with all c_k equal to one) converges absolutely for $|z| < 1$ to the function $\frac{1}{1-z}$. The convergence is uniform on any set of the form $\{z \in \mathbb{C} : |z| \leq r < 1\}$.

Theorem 7.26: **Radius of Convergence**

Any power series $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ has a radius of convergence R, with $0 \leq R \leq \infty$, such that

(a) if $r < R$, then $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ converges absolutely and uniformly on the closed disk $\overline{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$.

(b) if $|z - z_0| > R$ then the sequence of terms $c_k (z - z_0)^k$ is unbounded and $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ diverges.

Corollary 7.27

Suppose the power series $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ has a radius of convergence R. Then the series represents a function that is continuous on the disk $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$.

Corollary 7.28

Suppose the power series $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ has a radius of convergence R and γ is a smooth curve in $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$. Then

$$\int_\gamma \sum_{k=0}^{\infty} c_k (z - z_0)^k \, dz = \sum_{k=0}^{\infty} c_k \int_\gamma (z - z_0)^k \, dz$$

In particular, if γ is closed, then $\int_\gamma \sum_{k=0}^{\infty} c_k (z - z_0)^k \, dz = 0$.

THEOREM 7.30: **Determining Radius of Convergence**

For the power series $\sum_{k=0}^{\infty} c_k(z-z_0)^k$, the radius of convergence R is given by

$$R = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| \qquad \text{or} \qquad R = \lim_{k\to\infty} \frac{1}{|c_k|^{1/k}}$$

These are the **ratio test** and **root test** for determining radius of convergence.

THEOREM 8.1

Suppose $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ has positive radius of convergence R. Then f is holomorphic in $D_R(z_0) = \{z \in \mathbb{C} : |z-z_0| < R\}$.

THEOREM 8.2

Suppose $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ has positive radius of convergence R. Then

$$f'(z) = \sum_{k=1}^{\infty} kc_k(z-z_0)^{k-1}$$

and the radius of convergence of this power series is also R.

COROLLARY 8.5: **Taylor Series Expansion**

Suppose $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ has positive radius of convergence. Then

$$c_k = \frac{f^{(k)}(z_0)}{k!}.$$

COROLLARY 8.6: **Uniqueness of Power Series**

If $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ and $\sum_{k=0}^{\infty} b_k(z-z_0)^k$ are two power series that both converge to the same function $f(z)$ on an open disk centered at z_0, then $c_k = b_k$ for all k.

Theorem 8.7

Suppose f is a function that is holomorphic in $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$. Then f can be represented in $D_R(z_0)$ as a power series centered at z_0 (with a radius of convergence of at least R):

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \qquad \text{with} \qquad c_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}}\, dw.$$

Here γ is any positively oriented, simple, closed, smooth curve in $D_R(z_0)$ for which z_0 is inside γ.

Corollary 8.8

Suppose f is holomorphic on the region G, $w \in G$ and γ is a positively oriented simple, closed, smooth, g-contractible curve such that z_0 is inside γ, Then

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{k+1}}\, dz.$$

Corollary 8.9: **Cauchy's Estimate**

Suppose f is holomorphic in $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$ and $|f| \leq M$ there. Then

$$\left| f^{(k)}(z) \right| \leq \frac{k!M}{R^k}$$

for all $z \in \overline{D}_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$.

Analytic Function

A function f is **analytic** at a point z_0 if f can be represented as a power series centered at z_0 and with radius $R > 0$. f is analytic in an open region G if it is analytic at every point of G.

Theorem 8.10: **Analytic "Equals" Holomorphic**

A function f is analytic at a point z_0 if and only if it is holomorphic at z_0; f is analytic on G if and only if it is holomorphic on G.

THEOREM 8.11: **Classification of Zeros**

Suppose f is a holomorphic function defined on an open set G and suppose that f has a zero at some point $z_0 \in G$. Then there are exactly two possibilities: Either

(a) f is identically zero on some open disk D centered at z_0, or

(b) there is a positive integer m and a holomorphic function g, defined on G, satisfying $f(z) = (z - z_0)^m g(z)$ for all $z \in G$, with $g(z_0) = 0$.

The integer m in the second case is uniquely determined by f and z_0, and is called the **multiplicity** of the zero of f at z_0.

THEOREM 8.12: **The Identity Principle**

Suppose f and g are holomorphic on the region G and $f(z_k) = g(z_k)$ for a sequence that converges to $w \in G$, with $z_k \neq w$ for any k. Then $f(z) = g(z)$ for all $z \in G$.

THEOREM 8.13: **Maximum-Modulus Theorem**

Suppose f is holomorphic and non-constant on the closure of a bounded region G. Then $|f|$ only attains its maximum on the boundary of G.

THEOREM 8.15: **Maximum-Modulus Theorem**

Suppose f is holomorphic and non-constant on the closure of a bounded region G. Then $|f|$ only attains its minimum on the boundary of G.

DEFINITION 8.16: **Laurent Series**

A **Laurent series** centered at z_0 is a series of the form $\displaystyle\sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$.

THEOREM 8.19

Suppose f is a function that is holomorphic in the annulus

$$A = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}.$$

Then f can be represented in A as a Laurent series centered at z_0:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k \qquad \text{with} \qquad c_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}}\, dw.$$

Here γ is any circle in A that is centered at z_0 (or any other closed, smooth path that is A-homotopic to such a circle).

Theorem 8.20

For a given function in a given region of convergence, the coefficients of the corresponding Laurent series are uniquely determined.

Definition 9.1: **Classification of Singularities**

If f is holomorphic in the punctured disk $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$ but not at $z = z_0$, then z_0 is an **isolated singularity** of f. The singularity z_0 is called

(a) a **removable singularity** if there is a function g, that is holomorphic in the set $\{z \in \mathbb{C} : |z - z_0| < R\}$, such that $f = g$ in the set $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$,

(b) a **pole** if $\lim\limits_{z \to z_0} |f(z)| = \infty$,

(c) an **essential singularity** if z_0 is neither removable or a pole.

Proposition 9.5

Suppose z_0 is an isolated singularity of f. Then

(a) z_0 is removable if and only if $\lim_{z \to z_0} (z - z_0) f(z) = 0$;

(b) z_0 is pole if and only if it is not removable and $\lim_{z \to z_0} (z - z_0)^{n+1} f(z) = 0$ for some positive integer n. The smallest possible such n is called the **order** of the pole.

Theorem 9.6; **The Casorati-Weierstrass Theorem**

If z_0 is an essential singularity of f and $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$, then any $w \in \mathbb{C}$ is arbitrarily close to a point in $f(D)$. That is, for any $w \in \mathbb{C}$ and any $\varepsilon > 0$ there exists a $z \in D$ such that $|w - f(z)| < \varepsilon$.

Proposition 9.7

Suppose z_0 is an isolated singularity of f having Laurent series $\sum\limits_{k=-\infty}^{\infty} c_k (z - z_0)^k$ that is valid in some set $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Then

(a) z_0 is removable if and only if there are no negative exponents (that is, the Laurent series is a power series);

(b) z_0 is pole if and only if there are finitely many negative exponents, and the order of the pole is the largest value of k such that $c_{-k} \neq 0$;

(c) z_0 is an essential singularity if and only if there are infinitely many negative exponents.