Mathematical Statistics

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5 Joint Probability Distributions

Performance Criteria:

5. Understand and work with joint distributions.

    (a) Create a joint distribution table for a discrete probability distribution of two random variables.

    (b) Give probabilities of events for a discrete probability distribution of two random variables.

    (c) Find marginal probabilities for a discrete joint distribution of two random variables.

    (d) Give the two marginal distributions for a discrete joint distribution of two random variables.

    (e) Give a conditional probability associated with a discrete joint distribution of two random variables.

    (f) Give a conditional probability distribution for a discrete joint distribution of two random variables.

    (g) Determine whether two random variables are independent.

    (h) Find the value of a constant for which a given function of two variables is a joint probability density function.

    (i) Find probabilities for a continuous joint distribution.

    (j) Give the two marginal distributions for a continuous joint distribution of two random variables.

    (k) Give a conditional probability distribution for a continuous joint distribution of two random variables.

    (l) Determine whether two continuous random variables are independent.

    (m) Find the expected value and covariance of a discrete joint probability distribution.

    (n) Find the expected value and covariance of a continuous joint probability distribution.

    (o) Calculate probabilities using a multinomial probability distribution or multivariate hypergeometric distribution.

In this chapter we study joint probability distributions, which arise when we are considering two random variables having the same sample space.

5.1 Discrete Joint Distributions

Performance Criteria:

5. (a) Create a joint distribution table for a discrete probability distribution of two random variables.
   (b) Give probabilities of events for a discrete probability distribution of two random variables.

The table below shows political registrations of 200 residents of the state of Montana.

		PARTY REGISTRATION		
		Dem	**Rep**	**Ind**
GENDER	**Male**	35	45	24
	Female	47	33	16

1. Consider the experiment of randomly selecting one person out of this group. What is the probability that they are

   (a) male? (b) Republican? (c) male and Republican?

   (d) male or Republican? (e) male, given that they are Republican?

   (f) Republican, given that they are male?

The above type of table is sometimes called a **two-way table**. Using terminology analogous to what we used when we have worked with *Excel*, we call each of the spaces occupied by the numbers in the table **cells**. The numbers in the cells are called **cell counts**. Note that each cell represents a particular gender/party registration combination.

Considering the experiment of randomly selecting an individual from this group, let us define two random variables X and Y. X assigns to any outcome (individual selected) a number representing their party registration (zero for Democrat, one for Republican, two for Independent). Y assigns to each outcome a zero or one depending on gender (zero for male, one for female). We can make a new table based on the values of the random variables, with the number in each cell being the probability that an individual who is randomly selected from the group has the gender and party registration that the cell represents.

		X		
		0	**1**	**2**
Y	**0**	0.175	0.225	0.120
	1	0.235	0.165	0.080

2. Give each of the following probabilities. Take $(X=x, Y=y)$ to mean $X=x$ *and* $Y=y$.

   (a) $P(X=1, Y=0)$ (b) $P(X=1)$ (c) $P(Y=1)$

   (d) $P(X=1 \mid Y=0)$ (e) $P(X=2 \text{ or } Y=1)$

We will now define a function f of two variables, called a **joint probability distribution** (in this case, a *discrete* joint probability distribution), by

$$f(x,y) = P(X=x, Y=y) \qquad \text{for} \qquad x=0,1,2, \quad y=0,1$$

Thus, $f(2,0)=0.120$, $f(1,1)=0.165$ and so on. At the risk of being a bit redundant, we now show a table of values for the function f:

		x		
	$f(x,y)$	0	1	2
y	0	0.175	0.225	0.120
	1	0.235	0.165	0.080

◇ **Example 5.1(a):** Give each of the probabilities from Exercise 2 in terms of the probability function f.

The only one that might need some explanation is $P(X=1 \mid Y=0)$, where we must divide the probability $P(X=1, Y=0)$ by $P(Y=0)$:

$$P(X=1 \mid Y=0) = \frac{P(X=1, Y=0)}{P(Y=0)} = \frac{f(1,0)}{f(0,0)+f(1,0)+f(2,0)}\,.$$

In the next section we will encounter a concept that will allow us to give a simpler expression for the denominator. Here are the other probabilities in terms of f:

$$P(X=1, Y=0) = f(1,0) \qquad\qquad P(X=1) = f(1,0)+f(1,1)$$

$$P(Y=1) = f(0,1)+f(1,1)+f(2,1)$$

$$P(X=2 \text{ or } Y=1) = f(2,0)+f(2,1)+f(0,1)+f(1,1)\,.$$

The concept from the next section will allow us to give a cleaner expression for this last probability as well.

3. Consider the experiment of rolling a single die (one half of a pair of dice).

   (a) Give the sample space.

   (b) Define a random variable X on the sample space by $X=0$ if the number rolled is odd, $X=1$ if the number rolled is even. Give $\text{Ran}(X)$.

(c) Define the random variable Y on the same sample space to be the number of letters in the spelling of the number rolled. For example, $Y(5) = 4$ because the word *five* has four letters. Give $\text{Ran}(Y)$.

(d) Create a table for the joint probability distribution $f(x, y)$. It should have the form of the table on the previous page, but give all probabilities as fractions.

We now make a formal definition, based on things that should be intuitively clear. First, though, we recall that for two sets A and B, we define the **Cartesian product** of A and B, denoted by $A \times B$, to be the set of all ordered pairs (x, y) for which x is in A and y is in B. Recall also that $\text{Ran}(X)$ is the range of X, and similarly for Y.

Discrete Joint Probability Distribution

Let X and Y be random variables *defined on the same discrete sample space*. A function f, defined on $\text{Ran}(X) \times \text{Ran}(Y)$, is a **joint probability distribution** if

1) $f(x, y) \geq 0$ for all $(x, y) \in \text{Ran}(X) \times \text{Ran}(Y)$,

2) $\displaystyle\sum_{\text{Ran}(X)} \sum_{\text{Ran}(Y)} f(x, y) = 1$,

3) $P(X = x, Y = y) = f(x, y)$ for all $(x, y) \in \text{Ran}(X) \times \text{Ran}(Y)$.

Although the following theorem may be obvious, we state it because of its importance when we introduce continuous joint distributions.

Theorem 5.1: Let f be a discrete joint probability distribution for the random variables X and Y, and let A be any subset of $\text{Ran}(X) \times \text{Ran}(Y)$. Then

$$P((X, Y) \in A) = \sum_{(x,y) \in A} \sum f(x, y) .$$

4. The following table gives the joint probability distribution function f for two discrete random variables X and Y.

		x			
	$f(x,y)$	0	1	2	3
y	0	$\frac{5}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
	1	$\frac{3}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
	2	$\frac{2}{24}$	$\frac{1}{24}$	0	0

Give each of the following probabilities of events, first as an expression in terms of f evaluated at specific values of x and y, then give a numerical answer, **as a fraction**. Take $(X = a, Y = b)$ to mean $X = a$ *and* $Y = b$, and take $P(X = a \mid Y = b)$ to mean the conditional probability of $X = a$ given that $Y = b$. **You might consider giving one of the probabilities in terms of its complementary event**.

(a) $P(X = 1, Y = 0)$ (b) $P(X = 1)$ (c) $P(X + Y \leq 1)$

(d) $P(X = 2 \text{ or } Y = 1)$ (e) $P(X + Y \leq 3)$ (f) $P(Y = 2|X = 0)$

5. (a) Suppose that three marbles are to be drawn *without replacement* from an urn containing three blue marbles, two red marbles and five yellow marbles. Let X be the number of blue marbles drawn and Y be the number of red marbles drawn. Give a table like the one above for the joint probability distribution, *giving all probabilities in fraction form.* It is suggested that you make all fractions have denominators of 720; check your answers by making sure that the sum of all the probabilities is one.

   (b) Repeat part (a) under the conditions that the marbles are drawn *with replacement.* Here I would suggest that you make the denominators of all the fractions 1000.

5.2 Discrete Marginal Distributions

Performance Criteria:

5. (c) Find marginal probabilities for a discrete joint distribution of two random variables.

   (d) Give the two marginal distributions for a discrete joint distribution of two random variables.

Consider again the table from the last section, but with probabilities in place of counts:

		PARTY REGISTRATION		
		Dem	**Rep**	**Ind**
GENDER	**Male**	0.175	0.225	0.120
	Female	0.235	0.165	0.080

This table gives us probabilities of specific party/gender combinations, but does not directly give probabilities such as whether a randomly selected individual will be Republican, or be male, and so on. In terms of the random variables X and Y, we are asking for $P(X = 1)$ and $P(Y = 0)$. It should be obvious that to compute probabilities involving only one of the random variables X or Y one needs to add up the values in each each column and row of the distribution table:

		X			
		0	**1**	**2**	
Y	**0**	0.175	0.225	0.120	0.520
	1	0.235	0.165	0.080	0.480
		0.410	0.390	0.200	1.000

These additional probabilities are called **marginal probabilities**. They give the values of two additional distributions g and h defined by

$$g(x) = P(X = x) \quad \text{for} \quad x = 0, 1, 2 \qquad \text{and} \qquad h(y) = P(Y = y) \quad \text{for} \quad y = 0, 1\,.$$

For this particular example, g and h have the distributions

x	0	1	2
$g(x)$	0.41	0.39	0.20

y	0	1
$h(y)$	0.52	0.48

The functions g and h are called the **marginal distributions** associated with the joint distribution f defined in the last section. The value of 1.000 in the lower right is the sum of the probabilities for both g and h - as it should be, the total probability for each of those marginal distributions is one. For any joint distribution f, we define g and h more formally as follows.

Marginal Distributions

Let f be a discrete joint probability distribution for the random variables X and Y. We define the **marginal distribution functions** g and h on $\mathrm{Ran}(X)$ and $\mathrm{Ran}(Y)$, respectively, by

$$g(x) = \sum_{\mathrm{Ran}(Y)} f(x,y) \qquad \text{and} \qquad h(y) = \sum_{\mathrm{Ran}(X)} f(x,y)\,.$$

Note carefully what these two formulas say. To compute the value of g for a particular x, we fix the value of x and add up all the values of $f(x,y)$ as y ranges over all possible values of the random variable Y. A similar process is carried out for finding values of h. For the example we have been looking at, this means that

$$g(x) = f(x,0) + f(x,1) \qquad \text{and} \qquad h(y) = f(0,y) + f(1,y) + f(2,y)\,.$$

We can perhaps see this more readily by adding the marginal probabilities to our table for the distribution function $f(x,y)$:

		x			
	$f(x,y)$	0	1	2	$h(y)$
y	0	0.175	0.225	0.120	0.520
	1	0.235	0.165	0.080	0.480
	$g(x)$	0.410	0.390	0.200	

◇ **Example 5.2(a):** Three marbles are drawn, without replacement, from an urn containing 3 blue, 2 yellow and 5 red marbles. Let X be the random variable that assigns to any outcome the number of red marbles drawn, and let Y assign to any outcome the number of yellow marbles drawn. The table for the joint probability distribution f is shown below. Determine the marginal distributions $g(x)$ and $h(y)$.

		x			
	$f(x,y)$	0	1	2	3
y	0	$\frac{1}{120}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{12}$
	1	$\frac{1}{20}$	$\frac{1}{4}$	$\frac{1}{6}$	0
	2	$\frac{1}{40}$	$\frac{1}{24}$	0	0

To get the marginal distributions, we simply sum up the rows and columns to obtain the following table:

		x				
	$f(x,y)$	0	1	2	3	$h(y)$
y	0	$\frac{1}{120}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{7}{15}$
	1	$\frac{1}{20}$	$\frac{1}{4}$	$\frac{1}{6}$	0	$\frac{7}{15}$
	2	$\frac{1}{40}$	$\frac{1}{24}$	0	0	$\frac{1}{15}$
	$g(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	1

From this we can see that the marginal distributions are

x	0	1	2	3
$g(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

y	0	1	2
$h(y)$	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

◇ **Example 5.2(b):** For the distribution from the previous example, give each of the following probabilities in terms as efficiently as possible in terms of f, g and h.

$$P(X=2) \qquad P(X=1 \mid Y=2) \qquad P(X=1 \text{ or } Y=2)$$

Using the definitions of f, g, h and conditional probability, along with the addition rule, we have

$$P(X=2)=g(2) \qquad P(X=1 \mid Y=2)=\frac{f(1,2)}{h(2)}$$

$$P(X=1 \text{ or } Y=2)=g(1)+h(2)-f(1,2)$$

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1. Add the marginal probabilities to your table from Exercise 3(d) of the previous section, then give the marginal distribution g of X and the marginal distribution h of Y.

2. Consider again the joint distribution for gender and political affiliation. Use to to give each of the following probabilities.

   (a) $P(X=2)$ in terms of f (b) $P(X=2)$ in terms of g or h

   (c) $P(X=1, Y=0)$ in terms of f

   (d) $P(Y=0 \mid X=1)$ in terms of f and g or h

   (e) $P(X=1 \text{ or } Y=0)$ in terms of f alone

   (f) $P(X=1 \text{ or } Y=0)$ in terms of f, g and h

3. The following table gives the joint probability distribution function f for two discrete random variables X and Y.

		x			
	$f(x, y)$	0	1	2	3
y	0	$\frac{5}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
	1	$\frac{3}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
	2	$\frac{2}{24}$	$\frac{1}{24}$	0	0

Give the two marginal distributions $g(x)$ and $h(y)$.

x : $\qquad\qquad$ y :

$g(x)$: $\qquad\qquad$ $h(y)$:

4. Continue using the joint distribution given by the table in the previous exercise. Give each of the following probabilities in terms of the marginal distributions g and h, and use the joint distribution f as well, when necessary. Then give a numerical value for each, as a fraction.

(a) $P(X = 2)$ $\qquad$ (b) (b) $P(Y = 2)$ $\qquad$ (c) $P(Y \leq 1)$

(d) $P(X = 2 \text{ or } Y = 1)$ $\qquad$ (e) $P(X = 3|Y = 1)$ $\qquad$ (f) $P(Y = 2|X = 0)$

3. Add the marginal probabilities to your table from Exercise 5(a) of the previous section, then give the marginal distribution g of X and the marginal distribution h of Y.

5.3 Discrete Conditional Distributions, Independent Random Variables

Performance Criteria:

5. (e) Give a conditional probability associated with a discrete joint distribution of two random variables.
   (f) Give a conditional probability distribution for a discrete joint distribution of two random variables.
   (g) Determine whether two random variables are independent.

Consider one last time the table for the Montanans:

		PARTY REGISTRATION		
		Dem	**Rep**	**Ind**
GENDER	**Male**	35	45	24
	Female	47	33	16

Suppose that we wish to determine the probability that a randomly selected male voter from this group would be a Democrat. Because there are $35 + 45 + 24 = 104$ males, the probability of selecting a Democrat from among them is $\frac{35}{104} = 0.337$. You'll recall that what we have computed here is a **conditional probability**, in this case the probability that one of the voters is a Democrat, *given that they are male*. Suppose that we wished to determine the same probability from the table of values for the joint distribution function $f(x, y)$:

		x			
	$f(x,y)$	0	1	2	$h(y)$
y	0	0.175	0.225	0.120	0.520
	1	0.235	0.165	0.080	0.480
	$g(x)$	0.410	0.390	0.200	

In this case we would compute the desired probability by taking $\dfrac{0.175}{0.520} = 0.337$. You can see here that the quotient that gives us the desired probability is

$$\frac{f(0,0)}{h(0)}$$

We would find the probability of a voter being Republican, given that they are male, to be $\dfrac{f(1,0)}{h(0)}$ and the probability that a voter is Independent, given that they are male, to be $\dfrac{f(2,0)}{h(0)}$. These three values are the probabilities of a probability distribution function that we will denote by $f(x \mid 0)$. This is defined more formally on the next page.

Discrete Conditional Distributions

Let f be a discrete joint probability distribution for the random variables X and Y with marginal distributions g and h. For a fixed $y \in \text{Ran}(Y)$ we define the **conditional distribution function** $v(\ * \mid y)$ on $\text{Ran}(X)$ by

$$v(x \mid y) = \frac{f(x,y)}{h(y)} \quad \text{if} \quad h(y) > 0\,.$$

Similarly, for a fixed $x \in \text{Ran}(X)$, if $g(x) > 0$ the conditional distribution function $w(\ * \mid x)$ is defined by

$$w(y \mid x) = \frac{f(x,y)}{g(x)}$$

for all $y \in \text{Ran}(Y)$

The purpose of writing $v(\ * \mid y)$ instead of $v(x \mid y)$ is to indicate that x is a variable, whereas y is fixed (for the purposes of the conditional distribution).

◇ **Example 5.3(a):** Find the conditional distributions $v(x \mid 0)$ and $w(y \mid 2)$ for the probability distribution from Example 5.2(a), shown below.

		x				
	$f(x,y)$	0	1	2	3	$h(y)$
y	0	$\frac{1}{120}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{7}{15}$
	1	$\frac{1}{20}$	$\frac{1}{4}$	$\frac{1}{6}$	0	$\frac{7}{15}$
	2	$\frac{1}{40}$	$\frac{1}{24}$	0	0	$\frac{1}{15}$
	$g(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	1

Here we have $v(0\mid 0) = \frac{1}{120}/\frac{7}{15} = \frac{1}{56}$, $v(1\mid 0) = \frac{1}{8}/\frac{7}{15} = \frac{15}{56}$, $v(2\mid 0) = \frac{1}{4}/\frac{7}{15} = \frac{15}{28}$ and $v(3\mid 0) = \frac{1}{12}/\frac{7}{15} = \frac{5}{28}$, so the function $v(x\mid 0)$ is given by

x	0	1	2	3
$v(x\mid 0)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{15}{28}$	$\frac{5}{28}$

Similar computations give us

y	0	1	2
$w(y\mid 2)$	$\frac{3}{5}$	$\frac{2}{5}$	0

You will recall that if two events A and B are independent, the occurrence of B makes it no more or less likely that A will occur. That is, $P(A|B) = P(A)$. This leads us to the definition that A and B are independent if, and only if, $P(A \cap B) = P(A)P(B)$. We define independent random variables in a similar manner.

Independent Random Variables

Let X and Y be discrete random variables with joint distribution function f and marginal distribution functions g and h, respectively. If

$$f(x, y) = g(x)h(y)$$

for all $x \in \mathrm{Ran}(X)$ and $y \in \mathrm{Ran}(Y)$, then we say that the random variables X and Y are independent.

◇ **Example 5.3(b):** Determine whether the random variables X and Y from the distribution given in Example 5.3(a) (the table for the distribution is reproduced below) are independent.

		x				
	$f(x,y)$	0	1	2	3	$h(y)$
y	0	$\frac{1}{120}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{7}{15}$
	1	$\frac{1}{20}$	$\frac{1}{4}$	$\frac{1}{6}$	0	$\frac{7}{15}$
	2	$\frac{1}{40}$	$\frac{1}{24}$	0	0	$\frac{1}{15}$
	$g(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	1

We see that $g(0)h(0) = \frac{7}{15}\frac{1}{12} = \frac{7}{180} \neq \frac{1}{120} = f(0,0)$, so the random variables X and Y are dependent.

1. The following table gives the joint probability distribution function f for two discrete random variables X and Y. Give the conditional distributions $v(x|0)$ and $w(y|3)$ in the manner shown in Example 5.3(a). Are the random variables X and Y independent?

		x			
	$f(x,y)$	0	1	2	3
y	0	$\frac{5}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
	1	$\frac{3}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
	2	$\frac{2}{24}$	$\frac{1}{24}$	0	0

5.4 Continuous Joint Probability Density Functions

Performance Criteria:

5. (h) Find the value of a constant for which a given function of two variables is a joint probability density function.

   (i) Find probabilities for a continuous joint density function.

As with the single random variable situation, when we can have two random variables that are continuous rather than discrete. In that case we again have a probability density function, but of two variables. And as in the single random variable case, evaluating such a function at a single point has no real meaning - it is only when we integrate values over some region in the $\mathbb{R}^2$ plane that we get a probability. That is, the probability has density, but no mass at any single point. Here is the definition of such a function:

Continuous Joint Probability Density Function

Let X and Y be continuous random variables defined on the same sample space. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is a **joint probability density function** if

1) $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$,

2) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$,

3) For any subset A of $\mathbb{R}^2$, $\displaystyle P[(X, Y) \in A] = \iint_A f(x, y)\, dx\, dy$

◇ **Example 5.4(a):** Find the value of c for which

$$f(x) = \begin{cases} cx^2y & \text{for } (x, y) \in [0, 1] \times [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

is a continuous joint probability density function.

It should be clear that f meets condition (1) above. We must choose c so that condition (2) is met. We see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_0^2\int_0^1 cx^2y\, dx\, dy =$$

$$c\int_0^2 \left[\tfrac{1}{3}x^3y\right]_0^1 dy = \tfrac{1}{3}c\int_0^2 y\, dy = \tfrac{1}{3}c\left[\tfrac{1}{2}y^2\right]_0^2 = \tfrac{2}{3}c\ ,$$

so c must be $\frac{3}{2}$.

◇ **Example 5.4(b):** For the continuous joint probability density function

$$f(x) = \begin{cases} \frac{3}{2}x^2y & \text{for } (x,y) \in [0,1]\times[0,2] \\ 0 & \text{otherwise} \end{cases}$$

of the previous example, find $P(\frac{1}{2} \le X \le 1, 0 \le Y \le 1)$.

Here we apply (3) from the definition, where A is the set $[\frac{1}{2},1]\times[0,1]$:

$$P(\tfrac{1}{2} \le X \le 1, 0 \le y \le 1) = \int_0^1 \int_{\frac{1}{2}}^1 \tfrac{3}{2}x^2y\,dx\,dy =$$

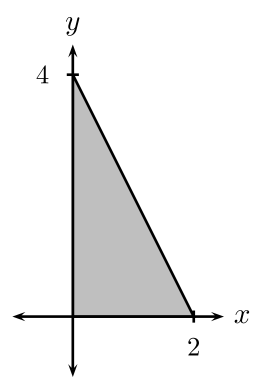
$$\int_0^1 \left[\tfrac{1}{2}x^3y\right]_{\frac{1}{2}}^1 dy = \int_0^1 \left[\tfrac{1}{2}y - \tfrac{1}{16}y\right] dy = \tfrac{7}{16}\int_0^1 y\,dy = \tfrac{7}{16}\left[\tfrac{1}{2}y^2\right]_0^1 = \tfrac{7}{32}$$

Although the integrals for both of the previous examples were computed "by hand," I would encourage you to use some technology like the **Wolfram Alpha**® *Double Integral Calculator* or your handheld calculator to make such computations.

◇ **Example 5.4(c):** Find $P(2X + Y \le 4)$ for the continuous joint probability density function

$$f(x) = \begin{cases} 2e^{-2x-y} & \text{for } x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}.$$

Setting up the iterated integral for the desired probability will be easier if we first determine the region in $\mathbb{R}^2$ where we are integrating. We first consider the equation $2x+y=4$, which is a line with x-intercept (found by setting $y=0$) two and y-intercept four. Because the point $(0,0)$ satisfies the inequality $2x+y\le 4$ the region of interest is the triangle shown to the right (because we only have nonzero probability in the first quadrant). We then see that the probability is given by either of the iterated integrals



$$\int_0^2 \int_0^{4-2x} 2e^{-2x-y}\,dy\,dx = \int_0^4 \int_0^{2-\frac{1}{2}y} 2e^{-2x-y}\,dx\,dy,$$

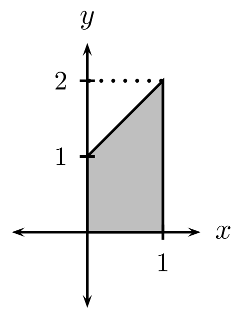
resulting in $P(2X+Y\le 4) = 1-5e^{-4} \approx 0.9084$.

◇ **Example 5.4(d):** For the continuous joint probability density function

$$f(x) = \begin{cases} \frac{3}{2}x^2y & \text{for } (x,y) \in [0,1]\times[0,2] \\ 0 & \text{otherwise} \end{cases},$$

find $P(Y \le X+1)$.

Because the density function is nonzero only on the rectangular region $[0,1]\times[0,2]$, we need only integrate over the portion of that region that satisfies the inequality $y \le x+1$. The graph of the line $y = x+1$ has slope one and y-intercept one and we are considering the points in the rectangle that are below that line, as shown in the graph to the right. Thus the desired probability is obtained by integrating over that region:



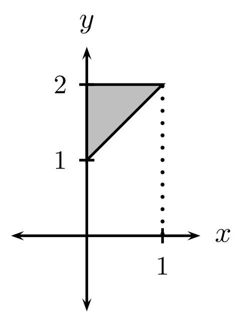
$$P(Y \le X+1) = \int_0^1 \int_0^{x+1} \tfrac{3}{2}x^2 y \; dy\, dx = \frac{31}{40}$$

We should make two observations regarding the previous exercise:

- If we were to change the order of integration, *two* iterated integrals would be required. This is because when y is a fixed value between zero and one we enter the region at $x = 0$ and leave at $x = 1$, but if y is a fixed value between one and two we enter at $x = y-1$ and leave at $x = 1$. Therefore we would have

$$P(Y \le X+1) = \int_0^1 \int_0^1 \tfrac{3}{2}x^2 y \; dx\, dy + \int_1^2 \int_{y-1}^1 \tfrac{3}{2}x^2 y \; dx\, dy$$

- Because the density function is nonzero only on the rectangular region $[0,1]\times[0,2]$, We could have instead obtained the desired probability by integrating over the triangle shown to the right and subtracting from one:



$$P(Y \le X+1) = 1 - \int_0^1 \int_{x+1}^2 \tfrac{3}{2}x^2 y \; dy\, dx = 1 - \frac{9}{40} = \frac{31}{40}$$

5.5 More on Continuous Joint Probability

Performance Criteria:

5. (j) Give the two marginal distributions for a continuous joint density function of two random variables.

   (k) Give a conditional probability distribution for a continuous joint density function of two random variables.

   (l) Determine whether two continuous random variables are independent.

Marginal Distributions

Let f be a continuous joint probability density function for the random variables X and Y. We define the **marginal distribution functions** $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$, respectively, by

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \qquad \text{and} \qquad h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

◇ **Example 5.5(a):** Find the marginal distributions g and h for the continuous joint probability density function

$$f(x) = \begin{cases} 6x^2y & \text{for } (x, y) \in [0, 1] \times [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

From the above definitions we have

$$g(x) = \int_0^1 6x^2y\, dy = \Big[3x^2y^2\Big]_0^1 = 3x^2 \quad \text{for } x \in [0, 1],\ 0 \text{ otherwise}$$

and

$$h(y) = \int_0^1 6x^2y\, dx = \Big[2x^3y\Big]_0^1 = 2y \quad \text{for } y \in [0, 1],\ 0 \text{ otherwise}$$

Conditional Distributions

Let f be a continuous joint probability density function for the random variables X and Y. For a fixed $y \in \mathbb{R}$ with $h(y) > 0$ we define the **conditional distribution function** $v(\ * \mid y) : \mathbb{R} \to \mathbb{R}$ by

$$v(x \mid y) = \frac{f(x,y)}{h(y)}$$

The notation $* \mid y$ is meant to indicate that y is fixed and the x is a variable.

Similarly, for a fixed $x \in \mathbb{R}$ with $g(x) > 0$ the conditional distribution function $w(\ * \mid x) : \mathbb{R} \to \mathbb{R}$ is defined by

$$w(y \mid x) = \frac{f(x,y)}{g(x)}$$

◇ **Example 5.5(b):** Find the conditional distributions $v(x \mid y)$ and $w(y \mid x)$ for the continuous joint probability density function

$$f(x) = \begin{cases} 6x^2 y & \text{for } (x,y) \in [0,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Applying the above definitions and using the results of Example 5.5(a) we have

$$v(x \mid y) = \frac{f(x,y)}{h(y)} = \frac{6x^2 y}{2y} = 3x^2 \quad \text{for } x \in [0,1],\ 0 \text{ otherwise}$$

and

$$w(y \mid x) = \frac{f(x,y)}{g(x)} = \frac{6x^2 y}{3x^2} = 2y \quad \text{for } y \in [0,1],\ 0 \text{ otherwise.}$$

We should make particular note of the results of this last example. Looking at the first part, we see that

$$v(x \mid y) = \frac{f(x,y)}{h(y)} = g(x).$$

This tells us that computing probabilities for the random variable Y for any fixed value of the random variable Y is the same as computing probabilities of X regardless of the value of Y; that is, X seems to be independent of Y. Multiplying both sides of the second equality by $h(y)$ results in

$$f(x,y) = g(x)h(y)$$

which, as in the case of two discrete random variables, we take to be the definition of independence:

Independent Random Variables

Let X and Y be continuous random variables with joint distribution function f and marginal distribution functions g and h, respectively. If

$$f(x,y) = g(x)h(y)$$

for all $(x,y) \in \mathbb{R}^2$, then we say that the random variables X and Y are statistically independent.

◇ **Example 5.5(c):** The continuous joint probability density function for two continuous random variables X and Y is

$$f(x) = \begin{cases} \frac{6}{5}(x^2+y) & \text{for } (x,y) \in [0,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

First we see that

$$g(x) = \int_0^1 \tfrac{6}{5}(x^2+y)\ dy = \tfrac{6}{5}\Big[x^2 y + \tfrac{1}{2}y^2\Big]_0^1 = \tfrac{6}{5}(x^2 + \tfrac{1}{2}) \quad \text{for } x \in [0,1],\ 0 \text{ otherwise}$$

and

$$h(y) = \int_0^1 \tfrac{6}{5}(x^2+y)\ dx = \tfrac{6}{5}\Big[\tfrac{1}{3}x^3 + xy\Big]_0^1 = \tfrac{6}{5}(y + \frac{1}{3}) \quad \text{for } y \in [0,1],\ 0 \text{ otherwise}$$

It looks doubtful that $f(x,y) = g(x)h(y)$, but I suppose we should check to be sure:

$$g(x)h(y) = \tfrac{6}{5}(x^2+\tfrac{1}{2}) \cdot \tfrac{6}{5}(y+\frac{1}{3}) = \tfrac{36}{25}(x^2 y + \tfrac{1}{3}x^2 + \tfrac{1}{2}y + \tfrac{1}{6}) \neq \tfrac{6}{5}(x^2+y)$$

Therefore the random variables X and Y are not independent.

Consider the following joint probability density function for two continuous random variables X and Y:

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{for } (x,y) \in [0,1] \times [0,2] \\ 0 & \text{otherwise} \end{cases}$$

1. Find $P(X \geq \frac{1}{2}, 0 \leq Y \leq \frac{3}{2})$. Give the integral used, and **evaluate it by hand, giving its value in fraction form**. Check your answer using a TI-89 or the Wolfram online double integral calculator (do a search for *online double integral calculator*.

2. Find $P(X+Y \geq 1)$ as follows: (a) Sketch the region over which the probability density function must be integrated to find the desired probability. (b) Set up the integral. (c) Evaluate the integral using a TI-89 or the Wolfram double integral calculator. You should get $\frac{7}{8}$; if you don't, try to correct your integral so that you do.

3. Find and simplify the marginal distribution $h(y)$, then use it to find $P(1 \le y \le 2)$.

4. Give the conditional distribution $f_{X|Y}(x|y)$ in simplified form.

5. Use your answer to (d) to find $P(X \le \frac{1}{2} \mid Y = 1)$. Do this by setting $y = 1$ and integrating over the desired range of x values.

5.6 Expected Value and Covariance of Joint Distributions

Performance Criteria:

5. (m) Find the expected value and covariance of a discrete joint probability distribution.

   (n) Find the expected value and covariance of a continuous joint probability distribution.

Expected Value of Joint Random Variables

Let X and Y be discrete random variables with joint distribution function f. The expected value of the distribution, denoted $\mu_{XY} = E(XY)$ is

$$E(XY) = \sum_{x \in \mathrm{Ran}(X)} \sum_{y \in \mathrm{Ran}(Y)} xyf(x,y)\,.$$

If X and Y are continuous random variables with joint distribution function f, the expected value $\mu_{XY} = E(XY)$ of the distribution is

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\; dx\, dy.$$

◇ **Example 5.6(a):** Find the expected value of the joint probability distribution, for discrete random variables X and Y, given in the table below.

		x		
	$f(x,y)$	0	1	2
y	0	0.175	0.225	0.120
	1	0.235	0.165	0.080

$$E(XY) = (0)(0)(0.175) + (1)(0)(0.225) + (2)(0)(0.120) + (0)(1)(0.235) + (1)(1)(0.165) + (2)(1)(0.080) = 0.325$$

◇ **Example 5.6(b):** Find the expected value of the joint probability distribution, for continuous random variables X and Y, given by

$$f(x) = \begin{cases} \frac{6}{5}(x^2+y) & \text{for } (x,y) \in [0,1]\times[0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\; dx\, dy = \int_0^1\int_0^1 xy\cdot \tfrac{6}{5}(x^2+y)\; dx\, dy =$$

$$\tfrac{6}{5}\int_0^1\int_0^1 (x^3y+xy^2)\; dx\, dy = \tfrac{6}{5}\int_0^1 \left[\tfrac{1}{4}x^4y+\tfrac{1}{2}x^2y^2\right]_0^1 dy =$$

$$\tfrac{6}{5}\int_0^1 (\tfrac{1}{4}y+\tfrac{1}{2}y^2)\; dy = \left[\tfrac{1}{8}y^2+\tfrac{1}{6}y^3\right]_0^1 = \tfrac{3}{16}$$

Covariance of Joint Random Variables

Let X and Y be discrete random variables with joint distribution function f. Let μ_X and μ_Y be the expected values of the marginal distributions $g(x)$ and $h(y)$. The **covariance** σ_{XY} of X and Y is

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_{x\in\mathrm{Ran}(X)}\sum_{y\in\mathrm{Ran}(Y)} (x-\mu_X)(y-\mu_Y)f(x,y)\,.$$

If X and Y are continuous random variables, then the covariance is given by

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y)f(x,y)\; dx\, dy.$$

As with the variance of a single random variable, the definition of the covariance of a joint distribution does not give the most efficient way to compute the covariance. Forthat we turn to the following result.

Theorem 5.2

Let X and Y be random variables (discrete or continuous) with joint distribution function f. The covariance of X and Y is given by

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y\,.$$

◇ **Example 5.6(c):** The joint probability distribution from Example 5.5(a) is given below, with the marginal distributions added. Find the covariance of the random variables X and Y.

		x			
	$f(x,y)$	0	1	2	$h(y)$
y	0	0.175	0.225	0.120	0.520
	1	0.235	0.165	0.080	0.480
	$g(x)$	0.410	0.390	0.200	

We can see that the expected values for X and Y are

$$\mu_X = (0)(0.41) + (1)(0.39) + (2)(0.20) = 0.79 \quad \text{and} \quad \mu_Y = (0)(0.52) + (1)(0.48) = 0.48$$

We computed the expected value in Example 5.5(a), so we can now compute

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = 0.325 - (0.79)(0.48) = -0.0542$$

---

1. The table below and to the right gives the joint probability distribution for two random variables X and Y. Use it for the following.

		x	
	$f(x,y)$	0	1
y	0	$\frac{5}{32}$	$\frac{15}{32}$
	1	$\frac{3}{32}$	$\frac{9}{32}$

   (a) Give $g(x)$ and $h(y)$ in table form, *with all values reduced.*

   (b) Compute $\displaystyle\sum_{x=0}^{1} x\,g(x)$. *Reduce your answer.*

   (c) Compute $\displaystyle\sum_{x=0}^{1}\sum_{y=0}^{1} x\,f(x,y)$; *reduce again.*

   (d) Your answers to (b) and (c) should be the same, and both are μ_X. μ_Y can be computed analogously, using either method. Find μ_Y.

   (e) Find the expected value $E(XY)$.

   (f) Find the covariance σ_{XY} using the formula $\sigma_{XY} = E(XY) - \mu_X\mu_y$.

   (g) Are X and Y independent? Show/explain. **This requires four mathematical statements!**

2. Give the expected value and covariance for the following joint probability density function for two continuous random variables X and Y:

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{for } (x,y) \in [0,1] \times [0,2] \\ 0 & \text{otherwise} \end{cases}$$

5.7 Multinomial and Multivariate Hypergeometric Distributions

Performance Criteria:

5. (o) Calculate probabilities using a multinomial probability distribution or multivariate hypergeometric distribution.

Consider an experiment with the following characteristics:

- It consists of n repeated trials.
- Each trial can result in any of k outcomes $E_1, E_2, E_3, ..., E_k$
- The probabilities of the outcomes of the outcomes, which we will denote $p_1, p_2, p_3, ..., p_k$, respectively, remain the same for each trial.
- The outcome of any single trial is independent of the outcomes of the previous trials.

We will refer to such an experiment as a **multinomial experiment**. Note that a Bernoulli experiment is simply a multinomial experiment for which $k = 2$.

Multinomial Probability

For a multinomial experiment with outcomes and probabilities as defined above, let $X_1, X_2, X_3, ..., X_k$ be random variables that assign the number of times the outcomes $E_1, E_2, E_3, ..., E_k$ occur, respectively. Then the joint probability distribution for these random variables is

$$f(x_1, x_2, x_3, ..., x_k; p_1, p_2, p_3, ..., p_k; n) = \binom{n}{x_1, x_2, ..., x_k} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}.$$

where

$$\binom{n}{x_1, x_2, ..., x_k} = \frac{n!}{x_1! x_2! \cdots x_k!}$$

It is instructive to compare this with the formula for the binomial distribution.

The multinomial distribution would apply to an experiment like drawing ten marbles, with replacement, from an urn containing 30 red marbles, 20 blue marbles, 40 yellow marbles and green marbles. In that case, the probability of getting two red, two blue, five yellow and one green marble is

$$f(2, 2, 5, 1; \frac{3}{10}, \frac{2}{10}, \frac{4}{10}, \frac{1}{10}; 10) = \frac{10!}{2!2!4!1!} \left(\frac{3}{10}\right)^2 \left(\frac{2}{10}\right)^2 \left(\frac{4}{10}\right)^5 \left(\frac{1}{10}\right)^1.$$

Suppose that we had the same urn, same number of draws, but we were drawing *without replacement*. You might guess that some variation of the hypergeometric distribution would be used, and you are correct. Note that the hypergeometric situation can be interpreted as follows: We are drawing from N objects with only two possible outcomes, with respective numbers k and

$N - k$ in the original urn. We draw n objects and want the probability of getting x of the first outcome and $n - x$ of the second outcome. The probability is given by

$$\frac{\begin{pmatrix} k \\ x \end{pmatrix}\begin{pmatrix} N-k \\ n-x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}$$

Lets change notation a bit by letting a_1 be the number of the N total objects that are of the first kind, and let a_2 be the number of objects of the second kind. Also, let x_1 and x_2 be the numbers of each of those kinds of objects that we ant the probability of obtaining when we draw n objects. Then the above expression becomes

$$\frac{\begin{pmatrix} a_1 \\ x_1 \end{pmatrix}\begin{pmatrix} a_2 \\ x_2 \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}$$

Now suppose we have N objects of k types, a_1 of type A_1, a_2 of type A_2, and so on through a_k of type A_k, and we are selecting n objects, without replacement.

Multivariate Hypergeometric Distribution

For an experiment as just described, let $X_1, X_2, X_3, ..., X_k$ be random variables that assign the number of objects of types $A_1, A_2, A_3, ..., A_k$ selected, respectively. Then the joint probability distribution for these random variables is

$$f(x_1, x_2, x_3, ..., x_k; a_1, a_2, a_3, ..., a_k, n, N) = \frac{\begin{pmatrix} a_1 \\ x_1 \end{pmatrix}\begin{pmatrix} a_2 \\ x_2 \end{pmatrix}\cdots\begin{pmatrix} a_k \\ x_k \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}.$$

H Solutions to Exercises

H.5 Chapter 5 Solutions

Section 5.1

1. (a) $\frac{35+45+24}{200} = \frac{104}{200} = \frac{13}{25}$ (b) $\frac{45+33}{200} = \frac{78}{200} = \frac{39}{100}$ (c) $\frac{45}{200} = \frac{9}{40}$

   (d) $\frac{35+45+24+33}{200} = \frac{137}{200}$ (e) $\frac{45}{45+33} = \frac{45}{78} = \frac{15}{23}$ (f) $\frac{45}{35+45+24} = \frac{45}{104}$

2. (a) $P(X = 1, Y = 0) = 0.225$ (b) $P(X = 1) = 0.225 + 0.165 = 0.390$

   (c) $P(Y = 1) = 0.235 + 0.165 + 0.080 = 0.480$

   (d) $$P(X = 1 \mid Y = 0) = \frac{0.225}{0.175 + 0.225 + 0.120} = \frac{0.225}{0.52} = 0.433$$

   (e) $P(X = 2 \text{ or } Y = 1) = 0.120 + 0.080 + 0.235 + 0.165 = 0.600$

3. (a) $S = \{1, 2, 3, 4, 5, 6\}$

   (b) $\text{Ran}(X) = \{0, 1\}$

   (c) $\text{Ran}(Y) = \{3, 4, 5\}$

   (d) See to the right.

		x		
	$f(x,y)$	0	1	2
y	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
	1	$\frac{2}{6}$	$\frac{1}{6}$	0

4. (a) $P(X = 1, Y = 0) = f(1, 0) = \frac{4}{24} = \frac{1}{6}$

   (b) $P(X = 1) = f(1, 0) + f(1, 1) + f(1, 2) = \frac{4}{24} + \frac{2}{24} + \frac{1}{24} = \frac{7}{24}$

   (c) $P(X + Y \leq 1) = f(0, 0 + f(1, 0) + f(0, 1) = \frac{5}{24} + \frac{4}{24} + \frac{3}{24} = \frac{12}{24} = \frac{1}{2}$

   (d) $P(X = 2 \text{ or } Y = 1) = f(2, 0) + f(2, 1) + f(2, 2) + f(0, 1) + f(1, 1) + f(3, 1) = \frac{10}{24} = \frac{5}{12}$

   (e) $P(X + Y \leq 3) = 1 - [f(2, 2) + f(3, 1) + f(3, 2)] = 1 - [0 + \frac{1}{24} + 0] = \frac{23}{24}$

   (f) $$P(Y = 2|X = 0) = \frac{f(0, 2)}{f(0, 0) + f(0, 1) + f(0, 2)} = \frac{\frac{2}{24}}{\frac{5}{24} + \frac{3}{24} + \frac{2}{24}} = \frac{2}{10} = \frac{1}{5}$$

5. (a)

		x			
	$f(x,y)$	0	1	2	3
y	0	$\frac{60}{720}$	$\frac{180}{720}$	$\frac{90}{720}$	$\frac{6}{720}$
	1	$\frac{120}{720}$	$\frac{180}{720}$	$\frac{36}{720}$	0
	2	$\frac{30}{720}$	$\frac{18}{720}$	0	0
	3	0	0	0	0

(b)

		x			
	$f(x,y)$	0	1	2	3
y	0	$\frac{125}{1000}$	$\frac{225}{1000}$	$\frac{135}{1000}$	$\frac{27}{1000}$
	1	$\frac{150}{1000}$	$\frac{180}{1000}$	$\frac{54}{1000}$	0
	2	$\frac{60}{1000}$	$\frac{36}{1000}$	0	0
	3	$\frac{8}{1000}$	0	0	0

Section 5.2

1. (a)

x	0	1	2
$g(x)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

(b)

y	0	1
$h(y)$	$\frac{3}{6}$	$\frac{3}{6}$

2. (a) $P(X=2) = f(2,0) + f(2,1)$ (b) $P(X=2) = g(2)$
   (c) $P(X=1, Y=0) = f(1,0)$
   (d) $P(Y=0 \mid X=1) = \frac{f(1,0)}{g(1)}$
   (e) $P(X=1 \text{ or } Y=0) = f(1,0) + f(1,1) + f(0,0) + f(2,0)$
   (f) $P(X=1 \text{ or } Y=0) = g(1) + h(0) - f(1,0)$

3.

x	0	1	2	3
$g(x)$	$\frac{10}{24}$	$\frac{7}{24}$	$\frac{4}{24}$	$\frac{3}{24}$

y	0	1	2
$h(y)$	$\frac{14}{24}$	$\frac{7}{24}$	$\frac{3}{24}$

4. (a) $P(X=2) = g(2) = \frac{4}{24} = \frac{1}{6}$ (b) (b) $P(Y=2) = h(2) = \frac{3}{24} = \frac{1}{8}$
   (c) $P(Y \leq 1) = h(0) + h(1) = \frac{14}{24} + \frac{7}{24} = \frac{21}{24} = \frac{7}{8}$
   (d) $P(X=2 \text{ or } Y=1) = g(2) + h(1) - f(2,1) = \frac{4}{24} + \frac{7}{24} - \frac{1}{24} = \frac{10}{24} = \frac{5}{12}$
   (e) $P(X=3|Y=1) = \dfrac{f(3,1)}{h(1)} = \dfrac{\frac{1}{24}}{\frac{7}{24}} = \dfrac{1}{7}$ (f) $P(Y=2|X=0) = \dfrac{f(0,2)}{g(0)} = \dfrac{\frac{2}{24}}{\frac{10}{24}} = \dfrac{1}{5}$

5. (a)

x	0	1	2	3
$g(x)$	$\frac{210}{720}$	$\frac{378}{720}$	$\frac{126}{720}$	$\frac{6}{720}$

(b)

y	0	1	2
$h(y)$	$\frac{336}{720}$	$\frac{336}{720}$	$\frac{48}{720}$

Section 5.3

1.

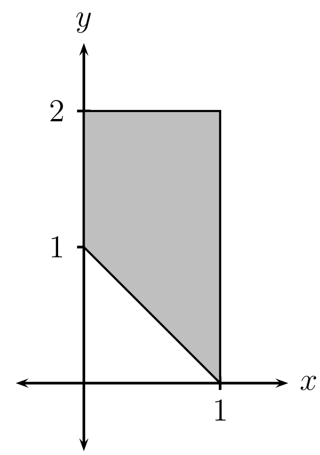
x	0	1	2	3
$v(x\|0)$	$\frac{5}{14}$	$\frac{4}{14}$	$\frac{3}{14}$	$\frac{2}{14}$

y	0	1	2
$w(y\|3)$	$\frac{2}{3}$	$\frac{1}{3}$	0

Because $f(0,0) = \frac{5}{24} \neq \frac{35}{144} = \frac{10}{24} \cdot \frac{14}{24} = g(0)h(0)$ X and Y are not independent.

Section 5.5

1. $P(X \geq \frac{1}{2}, 0 \leq Y \leq \frac{3}{2}) = \frac{1}{4}\int_0^{\frac{3}{2}}\int_{\frac{1}{2}}^1 (2x+y)\,dx\,dy = \frac{27}{64}$



2. (a) See graph to right.

   (b) $P(X+Y \geq 1) = \int_0^1 \int_{1-x}^2 \frac{1}{4}(2x+y)\,dy\,dx$

3. $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{1}{4}(2x+y)\,dx = \frac{1}{4}(1+y)$,

   $P(1 \leq Y \leq 2) = \int_1^2 h(y)\,dy = \frac{5}{8}$

4. (a) $v(x\,|\,y) = \dfrac{f(x,y)}{h(y)} = \dfrac{2x+y}{1+y}$

   (b) $P(X \leq \frac{1}{2} \mid Y = 1) = \int_0^{\frac{1}{2}} v(x\,|\,1)\,dx = \frac{3}{8}$

Section 5.6

1. (a)

x	0	1
$g(x)$	$\frac{1}{4}$	$\frac{3}{4}$

y	0	1
$h(y)$	$\frac{5}{8}$	$\frac{3}{8}$

   (b) $\sum_{x=0}^{1} x\,g(x) = \frac{3}{4}$ (c) $\sum_{x=0}^{1}\sum_{y=0}^{1} x\,f(x,y) = \frac{3}{4}$ (d) $\mu_Y = \sum_{y=0}^{1} y\,h(y) = \frac{3}{8}$

   (e) $E(XY) = \frac{9}{32}$ (f) $\sigma_{XY} = 0$

   (g) $f(0,0) = \frac{5}{32} = \frac{5}{8}\cdot\frac{1}{4} = g(0)h(0)$, $f(1,0) = \frac{15}{32} = \frac{5}{8}\cdot\frac{3}{4} = g(1)h(0)$,

   $f(0,1) = \frac{3}{32} = \frac{1}{4}\cdot\frac{3}{8} = g(0)h(1)$, $f(1,1) = \frac{9}{32} = \frac{3}{4}\cdot\frac{3}{8} = g(1)h(1)$

2. $g(x) = x + \frac{1}{2}$ for $x \in [0,1]$, 0 elsewhere, $h(y) = \frac{1}{4}y + \frac{1}{4}$ for $y \in [0,2]$, 0 elsewhere

   $\mu_X = E(X) = \frac{7}{12}$, $\mu_Y = E(Y) = \frac{7}{6}$, $\sigma_{XY} = E(XY) - \mu_x\mu_y = -\frac{1}{72}$