We propose a novel difference metric, called the graph diffusion distance (GDD), for quantifying the difference between two weighted graphs with the same number of vertices.

- Our metric is based on measuring the average similarity of heat diffusion on each graph by means of the graph Laplacian exponential kernel.
- The GDD is defined as the Frobenius norm of the difference of the diffusion kernels, at the diffusion time yielding the maximum difference.

**Proposition**

The GDD is a metric in the strict mathematical sense, i.e.

1. For any \( N \times N \) adjacency matrices \( A, B, C \)
   \( d_{gdd}(A, B, C) = 0 \) if \( A = B \)
   \( d_{gdd}(A, B, C) = d_{gdd}(A, B) + d_{gdd}(B, C) \) if \( d_{gdd}(A, B) \leq d_{gdd}(A, B) + d_{gdd}(B, C) \)

**Graph Diffusion Distance**

Let \( A_1 \) and \( A_2 \) be weighted adjacency matrices for \( N \) vertices, so both \( A_1 \) and \( A_2 \) are symmetric, non-negative, \( N \times N \) real matrices with zeros along the principle diagonal. The Edge Difference Distance (EDD) is defined as

\[
d_{edd}(A_1, A_2) = \| A_1 - A_2 \|_F.
\]

- The (unnormalized) graph Laplacian operator is defined according to \([3]: \) \( L_n = D_n - A_n \) for \( n = 1, 2 \), where \( D_n \) is a diagonal degree matrix for the adjacency \( A_n \), i.e. \( (D_n)_{i,i} = \sum_{j=1}^{N} (A_n)_{i,j} \).
- Given two graphs represented by \( L_1 \) and \( L_2 \), the Laplacian exponential kernels are defined as \( \exp(-tL_1) \) and \( \exp(-tL_2) \).

- The GDD

\[
\xi_{gdd}(A_1, A_2; t) = \| \exp(-tL_1) - \exp(-tL_2) \|_F
\]

\[
d_{gdd}(A_1, A_2) = \max \limits_{\xi_{gdd}(A_1, A_2; t)} \| \xi_{gdd}(A_1, A_2; t) \|_F
\]

where \( \| \cdot \|_F \) is the matrix Frobenius norm.

**Motivation**

The motivating principle behind our approach is the idea that two weighted graphs are similar if they enable similar patterns of information transmission.

**Proposition**

The GDD is a metric in the strict mathematical sense, i.e.

1. For any \( N \times N \) adjacency matrices \( A, B, C \)
   \( d_{gdd}(A, B, C) = 0 \) if \( A = B \)
   \( d_{gdd}(A, B, C) = d_{gdd}(A, B) + d_{gdd}(B, C) \) if \( d_{gdd}(A, B) \leq d_{gdd}(A, B) + d_{gdd}(B, C) \)

**Brain Connectivity Graphs**

Brain connectivity graphs are generated from diffusion MRI data using the following steps:

- In each voxel, a fiber orientation distribution (FOD) function is fit to the data using the method in [2].
- Axonal directions are extracted from the FOD by means of the tensor decomposition approach [1].
- A deterministic fiber tracking algorithm is used to integrate the axonal directions and generate brain connectivity map.
- We generate a brain connectivity graph using the procedure described in [4].

**Undersampling Experiment**

- We create subsets of the fully sampled by successively reducing the number of measurements.
- For each subset, we reconstruct a brain connectivity graph.
- The reconstructed graphs are used for comparison between EDD and GDD.

**References**


**Acknowledgements**

This work was funded in part by the NIH/NCRR Center for Integrative Biomedical Computing, P41RR12553